

ANALYTICAL DETERMINATION OF THE BURMESTER
LINES FOR CYLINDRIC-CYLINDRIC CRANKS

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1972

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Submitted to the Faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
for the Degree of
DOCTOR OF PHILOSOPHY
May, 1977

Thesis
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ACKNOWLEDGEMENTS

I wish to take the opportunity to express my sincere gratitude to all who have helped in any way to make this study possible. In particular, I am grateful to Dr. A. H. Soni, chairman of my advisory committee, for his continuous concern in the success of my research; to my other advisory committee members, Dr. D. Grace, for his encouragement and helpful suggestions, Dr. L. Fila and Dr. R. Lowery for their counsel and guidance. I am also grateful to Dr. Ferdinand Freudenstein, whose suggestions greatly contributed to defining the objectives and extent of this study.

The success of this study was in part due to Dr. R. Gumm, director of the UCC, Mr. J. McGee, Supervisor of Operations, and the fine team of computer operators of the Oklahoma State University Computer Center; to the Bank of Oklahoma in Tulsa, for permitting me to use their computer facilities in the preliminary phases of my research; to Dr. J. C. Thompson and Mr. M. O'Dell from the Department of Information and Computing Sciences of the University of Oklahoma, who permitted me to use their algebraic manipulation routine ALFRED.

I am fortunate to have met a fine group of "mechanisms men". To Dr. Dilip Kohli, Dr. In-Ping Lee, Professor Louis Torfason, Mr. Avinash Patwardhan, Mr. M. Siddhanty, Mr. M. S. Maiya, and Mr. A. S. Asif I offer my gratitude.

I would like to express my appreciation to Miss Lynda Baker and her

family for their warmth and continuous encouragement.

My special appreciation is due to my family, in particular to my dear stepfather, Dr. Attila Zadori, who passed away the same day I took my qualifying examination.

I am thankful to Ms. Mary Lehman for her help in typing the final manuscript.

Finally, I wish to thank the "Instituto Venezolano de Investigaciones Cientificas" for their financial support.

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LIST OF SYMBOLS

Σ	Moving rigid body
Σ'	Fixed rigid body
$\$_{ij}$	The screw that displaces a rigid body from its i th to j th position
θ_{ij}	Angle of rotation about screw $\$_{ij}$
d_{ij}	Translation along the screw $\$_{ij}$
(a_{ij}, b_{ij}, c_{ij})	Coordinates of a point on screw $\$_{ij}$ with respect to the X,Y,Z frame of reference
(u_{ij}, v_{ij}, w_{ij})	Direction cosines of a unit vector \bar{U}_{ij} parallel to screw $\$_{ij}$ with respect to the X,Y,Z frame of reference
$n_{i-1}, i = 2, \dots, 5$	$\tan \frac{\theta_{1i}}{2}, i = 2, \dots, 5$
R	Revolute pair
P	Prismatic pair
H	Helical pair
C	Cylindric pair
S	Spherical pair
(x_B, y_B, z_B)	Coordinates of a point on the Burmester line with respect to a x,y,z frame of reference
(ℓ, m, n)	Direction cosines of a unit vector parallel to the Burmester line with respect to a x,y,z frame of reference
s and Φ	Distance and twist angle between a Burmester line in the moving rigid body and its corresponding Burmester line in the fixed rigid body

CHAPTER I

INTRODUCTION

The motion of a rigid body, Σ , through several finitely separated positions relative to another rigid body, Σ' , is one of the most interesting and useful studies in the field of kinematics. Throughout the years this study has been referred to as the rigid body guidance problem.

The solution to the rigid body guidance problem becomes more complicated as the number of finitely separated positions of the rigid body, Σ , increases. In plane kinematics, for two or three positions of the rigid body, the solution for the rigid body guidance problem is fairly simple since it involves only linear expressions. For these cases, there are an infinite number of cranks which will displace a rigid body through two or three finitely separated positions.

For four finitely separated positions of a rigid body, Σ , executing planar motion, the rigid body guidance problem becomes complicated. In order to obtain the possible solutions, one must find a cubic curve¹ embedded in the moving frame of reference (Σ), and another cubic curve² embedded in the fixed³ frame of reference (Σ').

¹ Circle-point curve.

² Center-point curve.

³ Which actually could be moving.

Then, for each point selected in one of the cubics there will be a corresponding point in the other one. In this case there are an infinite number of cranks that will displace a rigid body through four finitely separated positions.

The number of cranks that can displace a rigid body through five finitely separated positions, however, is finite. There are a maximum of four for planar motion, and a maximum of six for spatial motion. To find these cranks, one may break up the five prescribed positions into two groups of four positions (e.g., positions 1, 2, 3 and 4, and positions 1, 2, 3 and 5, or any other two such combinations), then by tracing the cubic curves (for space motion, the cubic curves will become cubic cones) corresponding to each one of the two groups, one can find the intersection points for plane motion (for space motion, the cubic cones will intersect in lines). The points that are not poles (screws in space motion) are the ones which can be replaced for cranks which connect these points (screws in space motion) to their corresponding ones in the other frame of reference. These cranks permit a rigid body to be displaced through the five finitely separated positions.

The study of rigid body guidance for four and five finitely separated positions as described above was first performed by Ludwig Burmester [1, 2]⁴. It is interesting to note that the basic Burmester theory was developed as he was trying to design an indicator mechanism for a steam engine in which the coupler point of a four-link mechanism was required to trace a straight line. An excellent account on the

⁴ Numbers in brackets refer to similarly numbered references in the bibliography.

Burmester theory is presented by Burmester himself in reference [2].

Burmester theory has been the subject of study of many mathematicians and kinematicians as is evidenced by a large number of technical papers and books written on the subject. For practical purposes, this theory may be broken up into two groups: planar Burmester theory, and spherical and spatial Burmester theory. In order to recognize the enormous opportunities still existent in spherical and spatial kinematics, it is necessary to review some of the most significant achievements in planar Burmester theory.

1.1. Planar Burmester Theory

In order to facilitate the review of the most significant achievements in planar Burmester theory, we have chosen to present them as follows:

1) Construction procedure and location of Burmester points.

For four positions of a rigid body, Σ_i , $i = 1, \dots, 4$, relative to another one, Σ' , there are six poles which completely describe all the relative positions of Σ to move through the prescribed positions.

These six poles are denoted by P_{12} , P_{13} , P_{14} , P_{23} , P_{24} and P_{34} .

According to the Burmester theorem⁵, all points on a circle-point curve subtend either equal or complementary angles on the opposite sides of the "opposing-pole quadrangle"⁶. Therefore, a circle-point curve can be drawn for any one of the four positions (Beyer [52]).

⁵ Reference [60].

⁶ The opposing pole quadrangle is formed by two poles and two mirror image opposing poles (neither of whose subscript numbers are the same as the diagonally placed poles).

There are three different opposing-pole quadrangles of which any one may be used to construct the circle-point curve. The graphical procedures leading to the circle-point curve as proposed by Burmester [2] and Alt [4] are described in depth by Hain [60] and Beyer [52]. Corresponding to four positions of a rigid body executing planar motion, there is one circle point curve, and corresponding to this circle-point curve, there is a center-point curve. The center-point curve is constructed using any one of the three possible opposing-pole quadrangles.

Alt [11] discovered that in addition to the six poles, the center-point curve also contains six Π points which are obtained as points of intersection of the opposite sides of the three pole quadrangles. This property may be used to sketch the center-point curve.

The Burmester curves may also be found analytically, especially if a higher degree of accuracy is desired. These curves were given in equation form by Beyer [15, 19], who used concepts of analytic geometry and algebra, and by Hackmüller [16, 17, 21], who used a complex-number approach. The rationale for the analytical derivation to locate Burmester points is similar to that used in graphical approaches.

For five positions of a rigid body, Σ_i , $i = 1, \dots, 5$, moving relative to another rigid body, Σ^1 , we simply consider two groups of four positions (say Σ_i , $i = 1, 2, 3, 4$ and Σ_j , $j = 1, 2, 3, 5$ or any other combination) and find the corresponding Burmester curves. Those intersections which are not poles of the two circle-point curves are the Burmester points. Corresponding Burmester points may be found by intersecting the two center-point curves. Each pair of corresponding Burmester points constitutes a Burmester point pair. If these Burmester

point pairs are replaced by a crank with pin joints at both ends, then the moving rigid body, Σ , is capable of going through the five specified positions relative to the fixed rigid body Σ' .

All the graphical as well as the analytical procedures derived above may be used to determine the Burmester points. In particular, Hackmüller's approach [16] is very sound, since by eliminating the cubic terms of the center-point curve he derived the equations of 10 conic sections. He found that the Burmester points as well as one of the poles lie on each one of the conics. This approach has been used widely because of its simplicity in dealing with conic sections instead of cubic curves [3, 26, 31, 32, 34, 38, 50, 51].

Because of the difficulty in accurately locating the Burmester curves, several authors [22, 25, 28, 33, 34, 37, 42] investigated the possibility of varying the statement of the design problem so that either one or both Burmester curves⁷ would degenerate⁸ in a geometrically simpler form, such as a circle and a straight line. Others [58, 83, 84, 86, 88, 90] identified regions of the plane in which the Burmester curves may or may not exist. Their results are useful in the sense that by introducing the design data to their diagrams, one can determine in a few trials if the Burmester points exist. Computational procedures and drafting apparatus, capable of drawing the Burmester curves and locating the Burmester points, were developed by several investigators

⁷ Circle-point and center-point curves.

⁸ For Σ_i , $i = 1, \dots, 5$ there are five possible circle-point curves. By carefully selecting an origin for the coordinate system, as well as taking advantage of the geometry of the particular design problem, one or more circle-point and center-point curves may degenerate into a circle and a straight line.

[36, 48, 49, 55, 61, 69, 77, 78].

The problem of finding the Burmester points directly, i. e., the real roots of some polynomial, etc., became a challenge to some researchers. Freudenstein and Sandor [45] derived the locations of the Burmester points using a complex number approach. They obtained a fifth degree polynomial (in which one root is always a pole, and the other roots, when they exist, are the Burmester points) using Hackmüller's conics [16]. The coefficients of this fifth degree polynomial are in a form convenient for computer-based determination of the Burmester points. Artobolevskii et al [40] and Cherkudinov [41] derived a fourth degree polynomial in cartesian coordinates. The real roots of this polynomial yield the Burmester points. Other important contributions in analytically determining the Burmester points include Bottema [53], Dijksman [74], and Chang [92]. Bottema considered all five positions of a rigid body simultaneously in order to avoid the derivation of the center-point curve. He introduced a set of new unknowns which lead to a system of linear equations and quadratic equations. Thus, Bottema's approach reveals that the Burmester points may be found as the intersection of two conics. Dijksman developed a new method to derive the ten Hackmüller conics [16]. He found that each Burmester point is a common intersection point of corresponding members of three pencils of circles which are obtained from the center-point curves. Chang formulated the equations for the Burmester curves with respect to three "basic" relative poles, instead of the well-known four opposite relative poles. Two sets of three "basic" relative poles can be used for determining the unknown location of the Burmester point pairs. Synthesis is accomplished by using the equations for the three relative

poles on an inversion linkage.

More recently, Waldron [95, 96, 97] discussed the problem that usually occurs when synthesizing a linkage using Burmester theory, i. e., the inability to insure that the synthesized linkage would pass through the prescribed positions in a specified order. His technique of solving this problem basically reduces down to introducing constraining conditions on mobility in the design equations.

2) Properties and special cases.

Table I, as prepared by Primrose, Freudenstein and Sandor [54], presents a summary of some of the most important properties [10, 20, 24, 34, 42, 46, 47, 52].

TABLE I

PROPERTIES OF BURMESTER-POINT PAIRS ASSOCIATED WITH FIVE COPLANAR POSITIONS (i,j,k,l,m) OF A PLANE (Primrose, Freudenstein and Sandor [54])

Case	Given relationship	Result
1	Circlepoint, K_1 , at infinity; centerpoint, M_1 , finite.	Centerpoints M_j, M_k, M_l collinear
2	Centerpoint, M_1 , at infinity; circlepoint, K_1 , is a finite straight-line point.	Circlepoints K_j, K_k, K_l collinear
3	Three centerpoints (circlepoints) collinear.	Fourth circlepoint (centerpoint) at infinity
4	Two poles, P_{rs}, P_{st} and two centerpoints, M_i, M_j concyclic.	Points M_k, M_l, P_{rt} collinear
5	Four distinct collinear or concyclic circlepoints or centerpoints.	Cannot occur, except in special cases, such as Cardanic motion or its kinematic inversion
6	Straight-line point, K_1 collinear with two circlepoints, K_2, K_3 .	Fourth circlepoint, K_4 , coincident with straight-line point, K_1
7	Three circlepoints (centerpoints) have order-number 6; i.e., they are 6-point circlepoints (centerpoints).	Fourth circlepoint (centerpoint) is likewise a six-point circlepoint (centerpoint)
8	r-point circlepoints (centerpoints), where r exceeds 6.	Can have no more than two real such circlepoints (centerpoints)
9	Symmetrical motion of the first kind.	Real circlepoints (in symmetry position) and real centerpoints on axis of symmetry, or symmetrical about it
10	Symmetrical motion of the second kind.	Either two real centerpoints on axis of symmetry, the two corresponding circlepoints being collinear with AB, or two centerpoints and their corresponding circlepoints imaginary
11	Two circlepoints coincident with I, J, the circular points at infinity.	Common singular focus of circlepoint curves is itself a circlepoint

Alt [13, 14], Volmer [46], and Freudenstein and Sandor [45] studied the special case when three of the five specified positions are parallel. For this case the center-point curve degenerates into a circle and a line at infinity, and the singular focus (the center of the circle) does not lie on the curve. Freudenstein and Sandor [45] also studied the case when three of the five specified positions are on a circle. They found that only two Burmester points are possible and the center-point curves degenerate into straight lines; since the straight lines may only intersect at one point, and the Burmester points occur in pairs, the two Burmester points are coincident.

3) Applications.

Circular Burmester theory deals strictly with the relative motion of two rigid bodies. Therefore, this theory may be applied to mechanisms with four links [27, 52, 60, 68, 73, 75, 85, 87, 91, 94] or more [29, 30, 39, 43, 45, 59, 79, 80, 81, 82, 89, 91, 93] as well as mechanisms with slider pairs [35, 76].

Problems such as function generation, rigid body guidance for all cases, etc. can be solved with Burmester theory. Direct application of this theory is found in coupler driven mechanisms [5, 23], glass manufacturing machines [5], sewing machinery [7, 19], textile machinery [8, 9, 12], and many other mechanisms.

4) Extensions of Burmester theory.

Freudenstein, Bottema and Koetsier [67] extended the classical, finite, circular theory by studying the loci of points of a moving rigid body in coplanar motion, whose six corresponding positions lie on a conic section. The authors found that the general conic-section curve is a tricircular algebraic curve of order 7 which passes through the 15

poles of the moving system. This curve can be determined by any three complementary-pole quadrilaterals (in the moving plane), any two of which have two indices in common. They also found that there are 16 conic-section Burmester points associated with 7 distinct positions of a moving plane. Conic sections studied include the general case, parabola, hyperbola and ellipse.

Kaufman and Sandor [70] described the bicycloidal crank. In their study, the authors showed some analogies between the motion of the bicycloidal crank and the four-bar coupler, but here, instead of link lengths, they size the gears to allow rigid body guidance through up to five finitely separated positions and point-guidance through up to nine finitely separated positions.

Sandor [56] demonstrated the existence of a cycloidal Burmester theory in planar kinematics. This new theory involves points of the moving plane in several positions lying on cycloidal curves. He generated these curves with a "cycloidal crank " composed of a fixed pivot, a center of the generating circle and a crank pin (the author calls these three points a Burmester Point Trio, BPT). It was demonstrated that by using two such cycloidal cranks to form a single degree of freedom geared six-bar mechanism, a coupler may be moved through four prescribed positions of the input crank. He then applied the results to the synthesis of a geared six-bar mechanism for coordinating the positions of the input link with the positions of a rigid body. For the seven position rigid body guidance problem, the number of BPT's becomes finite. The design situation corresponds with the five positions of the classical Burmester theory.

1.2. Spherical and Spatial Burmester Theory

The planar Burmester theory lends itself for its possible extension to study spherical and space kinematics. In this section we will examine the key contributions describing in a progressive manner the development of some of the key concepts lending to space Burmester theory.

Four finitely separated positions of a rigid body, Σ , moving relative to another rigid body, Σ' , were studied in a landmark paper by Wilson [57]. He developed an analytical procedure which results in equations of nontranscendental form. By repeating his equations for four positions, he obtained a spatial circle-point curve, and a spatial center-point curve. His procedure is limited to Sphere-Sphere, Revolute-Sphere, Sphere-Revolute, and Revolute-Revolute cranks.

Roth [64] studied a rigid body Σ in a series of finitely separated positions in order to determine those points which lie on a sphere, circle, plane, line or cylinder. He used screw theory and linear transformations to describe rigid body motion in space. These linear transformations⁹ can be used to satisfy the conditions for points to lie on a sphere, circle, plane, line and cylinder. Roth applied these results for the synthesis of mechanisms [63]. He developed what he called similarity transformations (determination of the screw by pure rotations about two axes, one embedded in Σ and the other one embedded in Σ'). He found that finite rotations (as well as general displacements) do commute. Parallel (plane) and intersecting (sphere) screws were

⁹ These transformations repeated for $j = 2, 3$ and 4 (four positions of a rigid body) result in a cubic cone.

studied as special cases. He discussed points on a circle for these cases. His applications are for very simple mechanisms; this is understandable, since the constraining conditions for more sophisticated mechanisms are extremely complicated.

Roth [62] studied the motion of a rigid body Σ moving relative to another rigid body Σ' for up to five positions. He extended the concepts of pole triangle and pole quadrangle into space. Applying the cubic cones derived in reference [64] (space analogs of the planar Burmester curves), Roth obtained the conditions for the location and axis direction of the lines on Σ and their corresponding lines on Σ' . By replacing a cylindric pair collinear with the line in Σ and another one collinear to the corresponding line in Σ' and connecting them rigidly, we obtain a Cylindric-Cylindric (C-C) crank which will displace a rigid body Σ through four finitely prescribed positions relative to another rigid body Σ' . In this case there are an infinite number of C-C cranks. For five finitely prescribed positions the number of C-C cranks are finite (six, four, two or none). These are found by intersecting the two cubic cones corresponding to two groups of four positions (just as in plane). These lines are the space analogs of the planar Burmester points.

Sandor [66] applied the quaternions to develop procedures for kinematic synthesis of space mechanisms. He represented the space mechanisms as general kinematic chains consisting of one or more loops of ball-jointed bar-slideball members. Sandor studied the spatial circle-point theory to study four positions of a point of a rigid body which lies on a circle. He verified Roth's [64] results, by showing that there can only be a maximum of four points on a circle in space.

Sandor and Bishop [71] presented a general method of spatial kinematic synthesis by means of a stretch-rotation tensor. Basically they developed a stretch-rotation tensor operator in a matrix form, which will perform displacements of a linkage; the links that form this linkage are ball-ended, and one ball of each link is free to slide along the link's centerline. Thus, each link is represented by a vector that may vary both in magnitude and orientation. The method can be applied to multi-loop linkages, and to special cases, such as when a stretch or a rotation of one or more components does not exist.

Bottema, Koetsier and Roth [72] described the procedure to find the smallest circle determined by three positions of a rigid body in space. They found that the minimum radius circle may arise in one of two ways: either the minimum circle is associated with a point which lies on a screw axis or it is associated with a more general point. Their results are applicable for the design of the smallest Sphere-Revolute crank which will displace a rigid body through three finitely separated positions.

Chen and Roth [101, 102] unified Roth's [63, 64] results for finitely and infinitesimally separated positions of a rigid body, Σ , moving relative to another rigid body, Σ' . Their results are summarized in Table II.

Tsai and Roth [103] studied the geometrical conditions for incompletely specified displacements; i.e., rigid body displacements which are described by less than six independent parameters. They found the screws associated with these displacements and gave the constraining conditions for the design of cranks.

TABLE II
DESIGN OF LINKS AND DYADS
(Chen and Roth [102])

Link or dyad	No. of design positions	Locus (or number) of points that satisfy link constraint
Slider-slider-sphere	3	All points
	4	3rd-order surface
	5	6th-order curve
	6	10 points
Slider-sphere	2	All points
	3	3rd-order curve
Sphere-sphere	4	All points
	5	4th-order surface
	6	10th-order curve
	7	20 points
Revolute-sphere	3	All points
	4	6th-order curve
Revolute-slider-sphere	4	All points
	5	5th-order surface
	6	16th-order curve
	7	42 points
Revolute-slider-sphere	4	All points
	5	5th-order surface
	6	16th-order curve
	7	42 points
Cylinder-sphere	3	All points
	4	4th-order surface
	5	11th-order curve
	6	26 points
Cylinder-cylinder	3	All lines
	4	Line congruence
	5	6 lines
	3	One unique line
Cylinder-revolute	3	One unique line
Revolute-revolute	3	24 lines

Roth [100], by using the same approach as in [62], derived the constraining equations for the types of cranks listed in Table III.

TABLE III
BINARY CRANKS CONTAINING R, P AND C PAIRS

Type of Crank	Max. No. Positions	Type of Crank	Max. No. Positions
C - C	5	C - P	2
R - C	3	P - R	2
C - R	3	R - P	2
R - R	3	P - P	1
P - C	2		

Tsai and Roth [104] rederived all the constraining equations for all the cranks in [100] by using the "equivalent screw triangle". They also derived equations for cranks involving Helical (H) and Spherical (S) pairs as shown in Table IV.

TABLE IV
BINARY CRANKS CONTAINING A H OR S PAIR
AND A R, P, H, C, OR S PAIR

Type of Crank	Max. No. Positions	Type of Crank	Max. No. Positions
H - H	3	C - S	8
H - C	4	S - S	7
C - H	4	S - R	4
H - S	5	R - S	4
S - H	5	P - S	3
H - R	3	S - P	3
R - H	3	P - H	2
S - C	8	H - P	2

In [105], Tsai and Roth studied the Revolute-Revolute crank in depth. By using the constraining equations derived in [104], they obtain a sixth degree polynomial where the coefficients are in explicit form and whose real roots give the direction cosines of lines in the moving frame of reference. By further manipulations Tsai and Roth found the locations of these lines, as well as the direction cosines and location of the lines in the fixed frame of reference. Then, for R - R cranks, they replaced the lines in the fixed frame of reference and their corresponding lines in the moving frame of reference with R pairs. They also proved that there are only two R - R cranks that will displace a rigid body through three finitely separated positions in space.

1.3. Present Study

A brief analysis of the literature in plane and space kinematics

reveals that the key concepts of space kinematics were developed by using the analogy between planar and space motion of a rigid body. For example, the concepts of pole triangle, pole quadrangle, center-point and circle-point curves, and Burmester points for circular Burmester theory in plane led Roth to propose and develop relationships for space screw triangle, space screw quadrangle, cubic cones, and intersection of cubic cones to obtain Burmester lines for spherical and space motion.

The Burmester theory in plane in general considers two types of pairs such as revolute and prism pairs. In studying space motion, however, one has the option of building binary cranks using kinematic pairs such as Revolute (R) pair, Prism (P) pair, Helical (H) pair, Cylinder (C) pair, Spherical (S) pair, Plane (PL) pair, etc. Thus leaving aside the direct extension of planar Burmester theory, (i.e., planar R - R crank to space R - R crank), we obtain a class of problems which would fall into a category of generalized Burmester theory for rigid body motion in space. This class of problems will then consider guiding a rigid body through its theoretically possible, finitely separated positions by using such binary cranks as R - H, R - S, R - C, C - C, etc. (See Table III and IV.) Since the cylinder pair executes relative screw motion with a variable pitch, mathematical conditions can be imposed on its relative motion so that the C pair functions as a R, P, or H pair. This analysis shows that the C - C crank to guide a rigid body through its finitely separated positions will serve as the most general crank and that the Burmester theory developed for the C - C crank will provide all the necessary information to yield other types of binary cranks having R, P, H, or C pairs. It is important to note that the Burmester theory for a crank having S pairs is a problem in a class by

itself. Such theories have not been fully developed as yet, even after Wilson's publication [57].

The existing literature has provided numerical procedures to obtain the Burmester lines of a C - C crank. These numerical procedures provide a designer with an immediate solution to his design problem; however, theoretical investigations similar to the ones in Table I, which relate location and number of Burmester lines to positions of a rigid body, are difficult to undertake with these procedures. This is mostly due to the difficulty in selecting appropriate guesses of the initial values of the mechanism parameters. In order to avoid these difficulties, we are required to derive an analytical expression in explicit form whose input data are the positions of the rigid body and whose real roots lead to the determination of the direction cosines and location of the Burmester lines.

The objectives of this dissertation are:

- (1) To determine the Burmester lines for Cylindric-Cylindric cranks by deriving a polynomial relationship whose real roots yield the ratios of two of the three direction cosines of the Burmester lines in the moving frame of reference.

- (2) Obtain the conditions by which a C - C crank degenerates into other cranks with R, P, H and C kinematic pairs.

The procedure to achieve the above objectives will involve the following steps:

- (1) Describe the displacement of a rigid body in space by means of a screw displacement.
- (2) Describe the screw displacements for five finitely separated positions of a rigid body in space.

- (3) Obtain 4 equivalent screw triangles involving screw S_{ij} corresponding to the five positions of the rigid body, screw \hat{F}_{ij} in the fixed rigid body, and screw \hat{M}_{ij} in the moving rigid body.
- (4) Eliminate F_{ij} (direction cosines of screw \hat{F}_{ij} in the fixed rigid body) by setting the two determinants, corresponding to the two cubic cones, equal to zero.
- (5) Obtain the intersections (at most nine) of two cubic cones.
- (6) Eliminate the screws S_{12} , S_{13} , and S_{23}' , which also appear as intersections of the two cubic cones, and derive a sixth degree polynomial whose real roots permit us to find the ratios of two of the three direction cosines of the moving Burmester lines.
- (7) Locate the Burmester lines in the moving frame of reference; find the direction cosines and locations of the Burmester lines in the fixed frame of reference.
- (8) Obtain the conditions under which a cylinder pair will function as a Revolute, Prism, or a Helical pair.

In Chapter II, the screw displacements of a rigid body are derived by using Rodrigues' formula. This is reference material written for the benefit of the reader (Step 1).

In Chapter III, the sixth degree polynomial, whose real roots give the direction cosines of the Burmester lines, corresponding to C - C cranks, in the moving frame of reference, is derived (Steps 2 - 7).

In Chapter IV, the conditions under which a C pair may function as a R, P, or H pair are given (Step 8).

Chapter V presents the summary and conclusions.

CHAPTER II

DERIVATION OF RODRIGUES' FORMULA FOR SCREW

DISPLACEMENTS OF A RIGID BODY

The objective of this chapter is to provide a brief review involving the derivation of the Rodrigues formula.

In order to study the motion of rigid bodies in space, it is convenient to consider a fixed frame of reference, Σ' , and a moving frame of reference, Σ , embedded in two rigid bodies moving relative to each other (actually both rigid bodies may be in motion).

The displacement of a rigid body as it moves from one position to another in 3-dimensions may be described by several well known techniques. We will briefly describe the technique that will be used in this investigation.

2.1. Screw Axis Geometry

The displacement of a rigid body from a position Σ_i to a position Σ_j , may be interpreted as a rotation about, and a translation along a given axis, see Figure 1. This displacement is the screw displacement. We may represent the screw as S_{ij} , the rotation angle about the screw as θ_{ij} , and the translation along the screw as d_{ij} . θ_{ij} and d_{ij} are the screw parameters, and their ratio d_{ij} / θ_{ij} is the pitch of the screw. The displacement of the rigid body from position Σ_i to position Σ_j may be accomplished by simultaneously rotating and translating

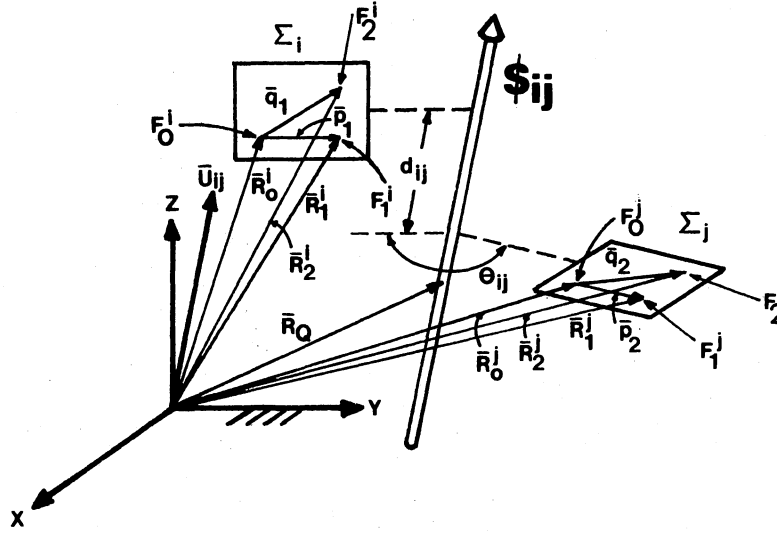


Figure 1. Screw Displacement

about and along the screw, or by considering the rotation and translation as occurring separately, and then superimposing their effects.

In order to locate a screw, we need to find a unit vector \bar{U}_{ij} in the fixed frame of reference with direction cosines (u_{ij}, v_{ij}, w_{ij}) which is parallel to the screw axis, and a point (a_{ij}, b_{ij}, c_{ij}) on the screw axis. This can be accomplished by using Rodrigues' formula.

2.2. Derivation of Rodrigues' Formula

Given any number of positions of a rigid body, Σ_i , $i = 1, \dots, n$, we can easily find all the screws that will displace it from one position to another by using Rodrigues' formula¹⁰ [98, 99].

In this study we consider five finite positions of a rigid body; thus we have altogether 10 possible screws; i. e., $\$_{12}$, $\$_{13}$, $\$_{14}$, $\$_{15}$, $\$_{23}$, $\$_{24}$, $\$_{25}$, $\$_{34}$, $\$_{35}$, and $\$_{45}$.

¹⁰ We use Rodrigues' formula to determine the screw parameters (since it is well known in the literature); however, the user may find the screws by any other approach.

In spatial kinematics, the screws, the cubic cones, and the Burmester lines are respectively the space analogs of the poles, the Burmester curves, and the Burmester points in plane kinematics. Roth showed that there exist in space screw-quadrangles which are the space analogs of the pole-quadrangles in plane. These screw-quadrangles can be completely defined by any four of the ten possible screws. In our analysis we use $\$_{12}$, $\$_{13}$, $\$_{14}$ and $\$_{15}$.

The general displacement of three non-collinear points in a rigid body, Σ , in its i th position (Σ_i) to its j th position (Σ_j) may be described by a screw displacement.

In Figure 1, we show three points (F_0^i, F_1^i, F_2^i) in Σ_i , and another three points (F_0^j, F_1^j, F_2^j) in Σ_j , which correspond to the displaced three points of Σ_i . Now, we connect the origin of coordinates (located in the fixed frame of reference) to each one of the points in Σ_i and Σ_j ; and define

$\bar{R}_0^i, \bar{R}_1^i, \bar{R}_2^i$ as the initial position vectors, and
 $\bar{R}_0^j, \bar{R}_1^j, \bar{R}_2^j$ as the displaced position vectors.

Let

$$\bar{R}_1^i - \bar{R}_0^i = \bar{p}_1 \quad (2.1)$$

$$\bar{R}_1^j - \bar{R}_0^j = \bar{p}_2 \quad (2.2)$$

$$\bar{R}_2^i - \bar{R}_0^i = \bar{q}_1 \quad (2.3)$$

$$\bar{R}_2^j - \bar{R}_0^j = \bar{q}_2 \quad (2.4)$$

Then, according to Rodrigues [99],

$$\bar{U}_{ij} \tan \frac{\theta_{ij}}{2} = \bar{W} = \frac{(\bar{p}_2 - \bar{p}_1) \times (\bar{q}_2 - \bar{q}_1)}{(\bar{q}_2 - \bar{q}_1) \cdot (\bar{p}_2 - \bar{p}_1)}, \quad (2.5)$$

where \bar{U}_{ij} is a unit vector which has direction cosines (u_{ij}, v_{ij}, w_{ij}) ,

$$(\bar{p}_2 + \bar{p}_1) \times (\bar{q}_2 + \bar{q}_1) \neq \bar{0}, \text{ and}$$

$$(\bar{p}_2 - \bar{p}_1) \times (\bar{q}_2 - \bar{q}_1) \neq \bar{0} \quad ,$$

we find that the translation of the rigid body is equal to the projection on \bar{W} of the vector connecting any one of the vertices of the triangle (formed by the three non-collinear points) before and after the displacement, thus

$$\bar{d}_{ij} = \frac{(\bar{R}_0^j - \bar{R}_0^i) \cdot \bar{W}}{|\bar{W}|^2} \bar{W} \quad . \quad (2.6)$$

If $(\bar{q}_2 - \bar{q}_1) \cdot (\bar{p}_2 - \bar{p}_1) = 0$ in Equation (2.5), then redefine the points with respect to another frame of reference.

Now, we need to find a point on a line parallel to the unit vector \bar{U}_{ij} with direction cosines (u_{ij}, v_{ij}, w_{ij}) to completely specify the screw axis.

Using \bar{i} , \bar{j} , and \bar{k} as unit vectors associated with the fixed frame of reference, the vectors \bar{R}_0^i and \bar{R}_0^j can be expressed as

$$\bar{R}_0^i = x_0^i \bar{i} + y_0^i \bar{j} + z_0^i \bar{k} \quad , \text{ and} \quad (2.7)$$

$$\bar{R}_0^j = x_0^j \bar{i} + y_0^j \bar{j} + z_0^j \bar{k} \quad , \text{ then} \quad (2.8)$$

the required point on the screw will be

$$\begin{bmatrix} x_Q \\ y_Q \\ z_Q \end{bmatrix} = \begin{bmatrix} x_0^i \\ y_0^i \\ z_0^i \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 & -w_{ij} \cot \frac{\theta_{ij}}{2} & v_{ij} \cot \frac{\theta_{ij}}{2} \\ w_{ij} \cot \frac{\theta_{ij}}{2} & 1 & -u_{ij} \cot \frac{\theta_{ij}}{2} \\ -v_{ij} \cot \frac{\theta_{ij}}{2} & u_{ij} \cot \frac{\theta_{ij}}{2} & 1 \end{bmatrix} \begin{bmatrix} x_0^j - x_0^i \\ y_0^j - y_0^i \\ z_0^j - z_0^i \end{bmatrix} \quad (2.9)$$

where (x_Q, y_Q, z_Q) are the coordinates of a point on the screw, and

$$\bar{R}_Q = x_Q \bar{i} + y_Q \bar{j} + z_Q \bar{k} \quad (2.10)$$

is the vector describing the position of this point.

In order to study the space Burmester theory for the C - C crank, Equations (2.5) and (2.9) will be employed extensively to obtain the screws given by the five finitely separated positions of a rigid body.

2.3. Screw Triangle Geometry

For three finitely separated positions of a moving rigid body in space, we can find by using Rodrigues' formula, three screws. Out of these three screws, only two are independent. By studying the geometry of the screw triangle (as given by Roth [62]) shown in Figure 2, we obtain analytical expressions for $\$_{ji}$ ¹¹ in terms of screws \hat{F}_{ij} and \hat{M}_{ij} .

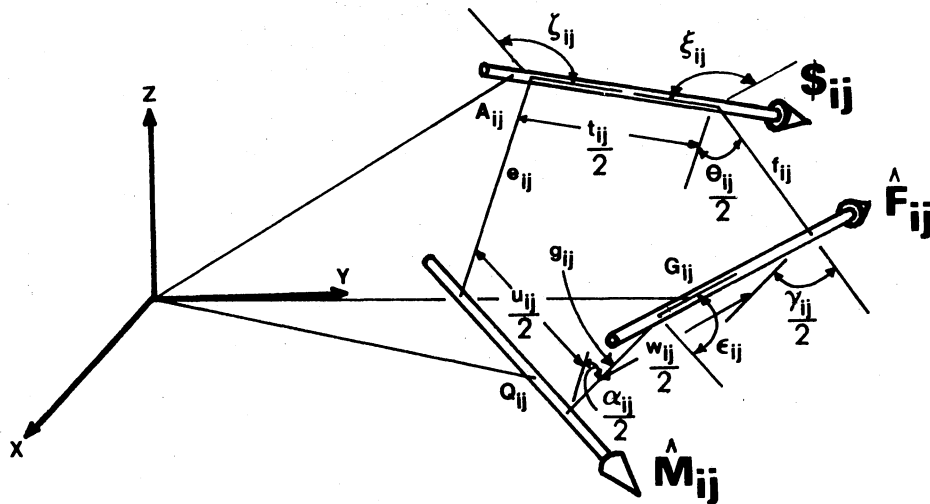


Figure 2. Screw Triangle Geometry

Taking projections along the screw axes yields:

¹¹ $\$_{ji}$ is a screw along the same axis and with the same magnitude as $\$_{ij}$; however, the sense of rotation and translation is opposite in both screws. $\$_{ji}$ is used instead of $\$_{ij}$ in order to preserve the symmetry, hence all the above expressions are valid under cyclic permutation of indices. The geometrical construction for determining a single screw equivalent to two successive screws is known as Halphen's theorem.

$$\frac{\theta_{ji}}{2} = \arcsin \left\{ \frac{\sin \epsilon_{ij}}{\sin \xi_{ij}} \sin \frac{\alpha_{ij}}{2} \right\} \quad (2.11)$$

$$\zeta_{ji} = \arcsin \left\{ -\frac{1}{\cos \frac{\alpha_{ij}}{2}} \left[\sin \epsilon_{ij} \cos \xi_{ij} + \cos \epsilon_{ij} \sin \xi_{ij} \cos \frac{\gamma_{ij}}{2} \right] \right\} ; \quad (2.12)$$

then,

$$\begin{aligned} t_{ji} = & \frac{2}{\sin \frac{\theta_{ji}}{2}} \left\{ \frac{u_{ij}}{2} \left[\sin \frac{\alpha_{ij}}{2} \cos \frac{\gamma_{ij}}{2} + \right. \right. \\ & \left. \left. + \cos \epsilon_{ij} \cos \frac{\alpha_{ij}}{2} \sin \frac{\gamma_{ij}}{2} \right] + \right. \\ & + \frac{w_{ij}}{2} \left[\cos \frac{\alpha_{ij}}{2} \sin \frac{\gamma_{ij}}{2} + \cos \epsilon_{ij} \sin \frac{\alpha_{ij}}{2} \cos \frac{\gamma_{ij}}{2} \right] - \\ & \left. - g_{ij} \sin \epsilon_{ij} \sin \frac{\alpha_{ij}}{2} \sin \frac{\gamma_{ij}}{2} \right\} \quad (2.13) \end{aligned}$$

$$\begin{aligned} e_{ji} = & \frac{1}{\sin^2 \frac{\theta_{ji}}{2}} \left\{ g_{ij} \left[\cos \frac{\gamma_{ij}}{2} \cos \frac{\theta_{ji}}{2} - \cos \frac{\alpha_{ij}}{2} \right] + \right. \\ & \left. + \sin \frac{\epsilon_{ij}}{2} \left[u_{ij} \sin \frac{\gamma_{ij}}{2} \cos \frac{\theta_{ji}}{2} - w_{ij} \sin \frac{\alpha_{ij}}{2} \right] \right\} . \quad (2.14) \end{aligned}$$

ϵ_{ij} and g_{ij} are, respectively, the angle and distance between \hat{M}_{ij} and \hat{F}_{ij} taken in the sense of screwing \hat{M}_{ij} about the line normal to both \hat{M}_{ij} and \hat{F}_{ij} into \hat{F}_{ij} . Similarly, for (ξ_{ij}, f_{ij}) and (ζ_{ij}, e_{ij}) we measure from \hat{F}_{ij} to ξ_{ji} and from ξ_{ji} to \hat{M}_{ij} , respectively.

CHAPTER III

DETERMINATION OF THE BURMESTER LINES

In Chapter I, we found that the procedures that exist for obtaining the Burmester lines in space are basically similar. They all require the finding of the intersections of two cubic cones. Due to the mathematical complexity, involved in the derivation and expansion of these cubic cones, the intersections were found numerically.

In a recent paper, Tsai and Roth [105] derived a sixth degree polynomial whose real roots give the Revolute-Revolute cranks for three finitely separated positions of a rigid body. Their procedure, together with the constraining conditions presented in reference [104] give a new light for the possibility of finding for the C - C cranks an explicit expression which will give the Burmester lines corresponding to five finitely separated positions of a rigid body.

The objectives of this chapter are to:

- 1) Study the geometry of the "equivalent screw triangle."
- 2) Transform the screws $\$_{1i}$, $i = 2, \dots, 5$ from a general coordinate system to one in which the z-axis coincides with $\$_{12}$ and the line perpendicular to $\$_{12}$ and $\$_{13}$ coincides with the y-axis.
- 3) Find the intersections of the two cubic cones.
- 4) Eliminate the screws $\$_{12}$, $\$_{13}$, and $\$_{23}'$, which are also intersections of the two cubic cones, and derive a sixth degree polynomial whose real roots yield the ratios of two of the three direction cosines of the

Burmester line in the moving frame of reference.

3.1. Equivalent Screw Triangle

Let S_{ij} represent a screw describing the finite displacement of a rigid body Σ from its i th to j th position. The rigid body moves relative to fixed rigid body Σ' . Let \hat{M}_{ij} represent the screw associated with the moving pair of a crank and \hat{F}_{ij} represent the screw associated with the fixed pair of the crank. Then, the three screws S_{ij} , \hat{M}_{ij} and \hat{F}_{ij} , as shown by Tsai, describe an equivalent screw triangle, see Figure 2.

The parameters of the screw triangles are θ_{ij} , α_{ij} , γ_{ij} and t_{ij} , u_{ij} , w_{ij} where θ_{ij} , α_{ij} , and γ_{ij} are the angles between the common normals between the screws S_{ij} , \hat{F}_{ij} , and \hat{M}_{ij} , and t_{ij} , u_{ij} and w_{ij} measure the distance along the axis of the screws. The relationships to calculate these parameters when S_{ij} , \hat{F}_{ij} and \hat{M}_{ij} are known are derived in reference [104]. These relationships are

$$\tan \frac{\theta_{ij}}{2} = \frac{-F_{ij} \cdot (S_{ij} \times M_{ij})}{(F_{ij} \times S_{ij}) \cdot (S_{ij} \times M_{ij})} \quad (3.1a)$$

$$\tan \frac{\alpha_{ij}}{2} = \frac{F_{ij} \cdot (S_{ij} \times M_{ij})}{(S_{ij} \times M_{ij}) \cdot (M_{ij} \times F_{ij})} \quad (3.1b)$$

$$\tan \frac{\gamma_{ij}}{2} = \frac{F_{ij} \cdot (S_{ij} \times M_{ij})}{(M_{ij} \times F_{ij}) \cdot (F_{ij} \times S_{ij})} \quad (3.1c)$$

$$\begin{aligned} \frac{t_{ij}}{2} = & - \frac{S_{ij} - (S_{ij} \cdot M_{ij}) M_{ij}}{1 - (S_{ij} \cdot M_{ij})^2} \cdot (Q_{ij} - A_{ij}) + \\ & + \frac{S_{ij} - (S_{ij} \cdot F_{ij}) F_{ij}}{1 - (S_{ij} \cdot F_{ij})^2} \cdot (G_{ij} - A_{ij}) \end{aligned} \quad (3.1d)$$

$$\begin{aligned} \frac{u_{ij}}{2} = & \frac{M_{ij} - (M_{ij} \cdot F_{ij}) F_{ij}}{1 - (M_{ij} \cdot F_{ij})^2} \cdot (G_{ij} - Q_{ij}) - \\ & - \frac{M_{ij} - (M_{ij} \cdot S_{ij}) S_{ij}}{1 - (M_{ij} \cdot S_{ij})^2} \cdot (A_{ij} - Q_{ij}) \end{aligned} \quad (3.1e)$$

$$\begin{aligned} \frac{w_{ij}}{2} = & \frac{F_{ij} - (F_{ij} \cdot S_{ij}) S_{ij}}{1 - (F_{ij} \cdot S_{ij})^2} \cdot (A_{ij} - G_{ij}) - \\ & - \frac{F_{ij} - (F_{ij} \cdot M_{ij}) M_{ij}}{1 - (F_{ij} \cdot M_{ij})^2} \cdot (Q_{ij} - G_{ij}) \end{aligned} \quad (3.1f)$$

$$u_{ij} = p_m \alpha_{ij} \quad (3.1g)$$

$$w_{ij} = p_f \gamma_{ij} \quad (3.1h)$$

where

S_{ij} = unit vector parallel to $\$_{ij}$

M_{ij} = unit vector parallel to \hat{M}_{ij}

F_{ij} = unit vector parallel to \hat{F}_{ij}

A_{ij} = a point on $\$_{ij}$

Q_{ij} = a point on \hat{M}_{ij}

G_{ij} = a point on \hat{F}_{ij}

p_m = pitch of \hat{M}_{ij}

p_f = pitch of \hat{F}_{ij}

For five positions of the rigid body Σ moving relative to Σ' , there are four independent screws, e. g., $\$_{12}$, $\$_{13}$, $\$_{14}$, and $\$_{15}$. These four screws together with the screws \hat{F}_{ij} in Σ' and \hat{M}_{ij} in Σ , yield at most four equivalent screw triangles.

By applying Equation (3.1a) for $j = 2, 3, 4, 5$ we obtain a relationship which is a polynomial whose roots are the ratios of the direction cosines of the Burmester lines in the moving rigid body .

The process of algebraic manipulation of the Equation (3.1a) written four times can be greatly simplified provided we establish a proper reference frame to locate in it the four equivalent screw triangles. For example, Tsai and Roth [105] forced the $\$_{12}$ to coincide with the z-axis and the line perpendicular to screws $\$_{12}$ and $\$_{13}$ to coincide with the y-axis.

Since we are dealing with a five position rigid body guidance problem, we are faced with four independent screws $\$_{12}$, $\$_{13}$, $\$_{14}$, and $\$_{15}$. One proper choice of establishing a suitable frame of reference is the same as that employed by Tsai and Roth. The next step is to locate the remaining screws $\$_{14}$ and $\$_{15}$ in this newly established frame of reference x,y,z. The transformation of coordinates, from a general coordinate system X,Y,Z to a new coordinate system x,y,z in which $\$_{12}$ coincides with the z-axis and the line normal to $\$_{12}$ and $\$_{13}$ coincides with the y-axis, is presented in the following section.

3.2. Coordinate Transformation

In Chapter II we showed how to find a screw, $\$_{ij}$, given two positions of a rigid body. In our study we will consider screws $\$_{12}$, $\$_{13}$, $\$_{14}$, and $\$_{15}$ which displace the rigid body from Σ_1 to Σ_2 , Σ_1 to Σ_3 , Σ_1 to Σ_4 , and Σ_1 to Σ_5 respectively. In this way, we can keep Σ_1 as the reference position.

Given five sets of three non-collinear points corresponding to five positions of the moving body, Σ , we can find screws $\$_{ij}$, $i = 1$, $j = 2, 3, 4, 5$ using Equations (2.1 - 2.10). The system of coordinates X,Y,Z can be transformed to a new system of coordinates x,y,z by forcing the z-axis to be coincident with $\$_{12}$, and the y-axis to be coincident with

the line which is perpendicular to both $\$_{12}$ and $\$_{13}$; see Figure 3.

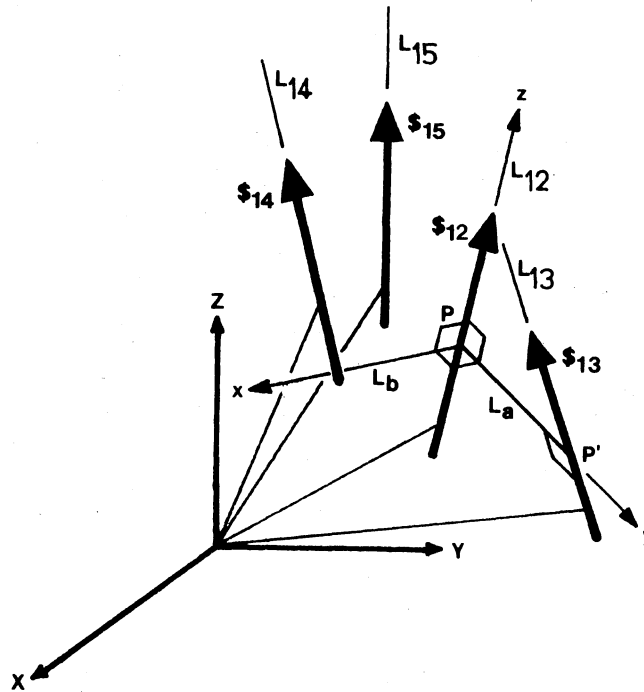


Figure 3. Transformation of Coordinates (X,Y,Z to x,y,z)

The transformation of coordinates is performed using the following steps:

- 1) Let L_{12} , L_{13} , L_{14} and L_{15} be lines which are collinear to $\$_{12}$, $\$_{13}$, $\$_{14}$ and $\$_{15}$ respectively. ($\$_{12}$ not parallel to $\$_{13}$).
- 2) Find a line L_a which is perpendicular to $\$_{12}$ and $\$_{13}$ at the same time.

For this purpose, we define P as the point of intersection between L_{12} and L_a . Similarly, P' is defined as the point of intersection between L_{13} and L_a . The coordinates of point P are $(a_{12}+u_{12}r, b_{12}+v_{12}r, c_{12}+w_{12}r)$ and the coordinates of point P' are $(a_{13}+u_{13}r', b_{13}+v_{13}r', c_{13}+w_{13}r')$ where r is proportional to the distance of P from point (a_{12}, b_{12}, c_{12}) , and r' is proportional to the distance of P' from point (a_{13}, b_{13}, c_{13}) .

b_{13}, c_{13}). Whence the direction cosines, of a line L_a passing through P and P', are proportional to $(a_{12}-a_{13}+u_{12}r-u_{13}r', b_{12}-b_{13}+v_{12}r-v_{13}r', c_{12}-c_{13}+w_{12}r-w_{13}r')$.

Since line L_a is at right angles to L_{12} and L_{13} , we have

$$u_{12} (a_{12}-a_{13}+u_{12}r-u_{13}r') + v_{12} (b_{12}-b_{13}+v_{12}r-v_{13}r') + w_{12} (c_{12}-c_{13}+w_{12}r-w_{13}r') = 0 \quad (3.2)$$

and

$$u_{13} (a_{12}-a_{13}+u_{12}r-u_{13}r') + v_{13} (b_{12}-b_{13}+v_{12}r-v_{13}r') + w_{13} (c_{12}-c_{13}+w_{12}r-w_{13}r') = 0 \quad (3.3)$$

Rewriting these two equations in terms of r and r' , we have

$$r - A r' + B = 0 \quad (3.4)$$

$$- A r - r' + C = 0 \quad (3.5)$$

where

$$A = u_{12}u_{13} + v_{12}v_{13} + w_{12}w_{13}$$

$$B = u_{12} (a_{12}-a_{13}) + v_{12} (b_{12}-b_{13}) + w_{12} (c_{12}-c_{13})$$

$$C = u_{13} (a_{12}-a_{13}) + v_{13} (b_{12}-b_{13}) + w_{13} (c_{12}-c_{13})$$

Solving Equations (3.4) and (3.5) for r and r' , we obtain

$$r = \frac{A C - B}{1 - A^2} \quad (3.6)$$

$$r' = A \frac{A C - B}{1 - A^2} + C \quad (3.7)$$

then the coordinates of points P and P' are:

$$P: (p_x, p_y, p_z) = (a_{12}+u_{12} \frac{AC-B}{1-A^2}, b_{12}+v_{12} \frac{AC-B}{1-A^2}, c_{12}+w_{12} \frac{AC-B}{1-A^2}) \quad (3.8)$$

$$P': (p'_x, p'_y, p'_z) = (a_{13}+u_{13} (A \frac{AC-B}{1-A^2} + C), b_{13}+v_{13} (A \frac{AC-B}{1-A^2} + C), c_{13}+w_{13} (A \frac{AC-B}{1-A^2} + C)) \quad (3.9)$$

Notice that point P is the origin of coordinates of the new coordinate system x, y, z .

The line L_a which passes through point P and P' is of the form

$$\frac{X - p_x}{p'_x - p_x} = \frac{Y - p_y}{p'_y - p_y} = \frac{Z - p_z}{p'_z - p_z} \quad (3.10)$$

or

$$\frac{X - p_x}{\ell_a} = \frac{Y - p_y}{m_a} = \frac{Z - p_z}{n_a} \quad (3.11)$$

where (ℓ_a, m_a, n_a) are the direction cosines of line L_a , and are defined as:

$$\ell_a = (p'_x - p_x) / s_{23}$$

$$m_a = (p'_y - p_y) / s_{23}$$

$$n_a = (p'_z - p_z) / s_{23}$$

$$s_{23} = \sqrt{(p'_x - p_x)^2 + (p'_y - p_y)^2 + (p'_z - p_z)^2},$$

here s_{23} is the distance between P and P' .

In the event that $\$_{12}$ intersects $\$_{13}$, we can find the direction cosines (ℓ_a, m_a, n_a) of line L_a from Appendix A, by using $\text{sgn} = 1$, and lines $L_{12} (u_{12}, v_{12}, w_{12})$ and $L_{13} (u_{13}, v_{13}, w_{13})$ as the intersecting lines.

3) Find a line L_b that is perpendicular to $\$_{12}$ and L_a .

We can find the direction cosines (ℓ_b, m_b, n_b) of line L_b from Appendix A, by using $\text{sgn} = -1$, and lines $L_{12} (u_{12}, v_{12}, w_{12})$ and $L_a (\ell_a, m_a, n_a)$ as the intersecting lines. Since we want line L_b to pass through point P with coordinates (p_x, p_y, p_z) , line L_b will be of the form

$$\frac{X - p_x}{\ell_b} = \frac{Y - p_y}{m_b} = \frac{Z - p_z}{n_b} \quad (3.12)$$

4) Determine the direction cosines of $\$_{12}$, $\$_{13}$, $\$_{14}$, and $\$_{15}$ in the transformed coordinate system.

Since we have forced the z -axis, of the transformed coordinate

system, to coincide with $\$_{12}$, the new direction cosines of this screw are (0,0,1). We also chose (p_x, p_y, p_z) as a point on $\$_{12}$ and as the origin of coordinates of the transformed coordinate system, therefore, the coordinates of this point relative to the new system are (0,0,0).

The direction cosines for $\$_{13}$, $\$_{14}$, and $\$_{15}$ in the transformed coordinate system are:

$$\begin{aligned}
 \$_{13} : \quad & u_{23} = u_{13}^{\ell} b + v_{13}^m b + w_{13}^n b \\
 & v_{23} = 0 \quad (\text{since } \$_{13} \text{ is perpendicular to } L_a) \\
 & w_{23} = u_{12} u_{13} + v_{12} v_{13} + w_{12} w_{13} \\
 \$_{14} : \quad & u_{24} = u_{14}^{\ell} b + v_{14}^m b + w_{14}^n b \\
 & v_{24} = u_{14}^{\ell} a + v_{14}^m a + w_{14}^n a \\
 & w_{24} = u_{12} u_{14} + v_{12} v_{14} + w_{12} w_{14} \\
 \$_{15} : \quad & u_{25} = u_{15}^{\ell} b + v_{15}^m b + w_{15}^n b \\
 & v_{25} = u_{15}^{\ell} a + v_{15}^m a + w_{15}^n a \\
 & w_{25} = u_{12} u_{15} + v_{12} v_{15} + w_{12} w_{15}
 \end{aligned}$$

3.3 Intersection of Two Cubic Cones

Once the transformation of coordinates has been completed, we are ready to locate the lines of intersection of two cubic cones, see Figure 4. First, we need to refer to Figure 2 and Equation (3.1a). In this equation, $M_{ij} = (M_x, M_y, M_z)$, $F_{ij} = (F_x, F_y, F_z)$ and $S_{ij} = S_{1j}$, $j = 2, \dots, 5$ which are the direction cosines of $\$_{1j}$, $j = 2, \dots, 5$.

For $j = 2$ in Equation (3.1a) we obtain

$$\begin{aligned}
 & [(F_x, F_y, F_z) \times (0, 0, 1)] \cdot [(0, 0, 1) \times (M_x, M_y, M_z)] \tan \frac{\theta_{12}}{2} + \\
 & + (F_x, F_y, F_z) \cdot [(0, 0, 1) \times (M_x, M_y, M_z)] = 0
 \end{aligned}$$

which simplifies into

$$(M_x \tan \frac{\theta_{12}}{2} + M_y) F_x + (M_y \tan \frac{\theta_{12}}{2} - M_x) F_y = 0 \quad .$$

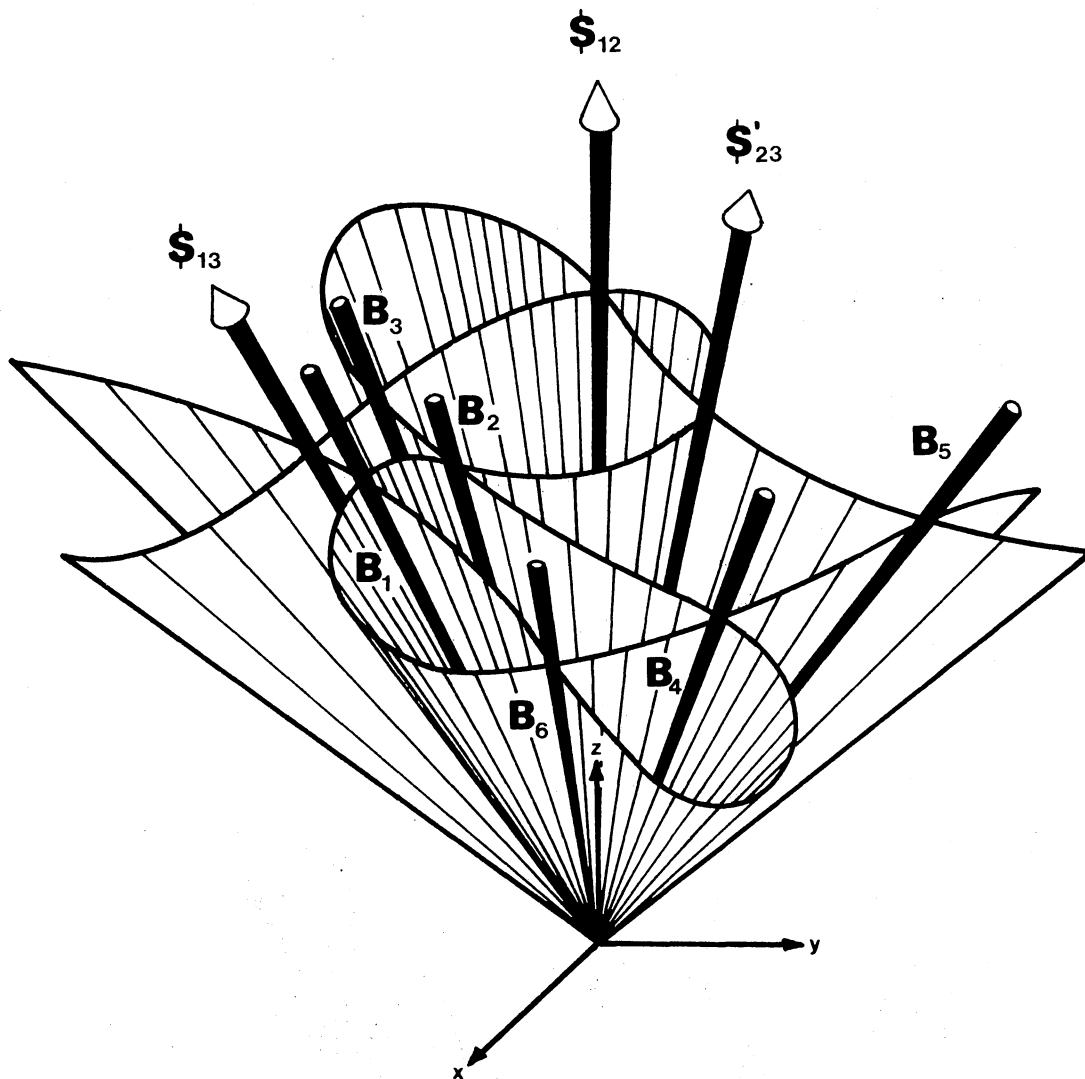


Figure 4. Intersections of Two Cubic Cones

Rewriting this equation and proceeding similarly for $j = 3, 4, 5$ we finally obtain the following expressions¹³:

$$a_1^F x + a_2^F y = 0 \quad (3.13)$$

$$a_3^F x + a_4^F y + a_5^F z = 0 \quad (3.14)$$

$$a_6^F x + a_7^F y + a_8^F z = 0 \quad (3.15)$$

$$a_9^F x + a_{10}^F y + a_{11}^F z = 0 \quad (3.16)$$

¹³ Provided not all the screws are parallel (planar case), and $\theta_{12} \neq 0$ or 180 degrees.

where

$$a_1 = M_x n_1 + M_y$$

$$a_2 = M_y n_1 - M_x$$

$$a_3 = -w_{23} (m_1 n_2 + M_y)$$

$$a_4 = m_1 - M_y n_2$$

$$a_5 = u_{23} (m_1 n_2 + M_y)$$

$$a_6 = (v_{24} m_2 + w_{24} m_3) n_3 + m_4$$

$$a_7 = (w_{24} m_4 - u_{24} m_2) n_3 - m_3$$

$$a_8 = -(u_{24} m_3 + v_{24} m_4) n_3 + m_2$$

$$a_9 = (v_{25} m_5 + w_{25} m_6) n_4 + m_7$$

$$a_{10} = (w_{25} m_7 - u_{25} m_5) n_4 - m_6$$

$$a_{11} = -(u_{25} m_6 + v_{25} m_7) n_4 + m_5$$

and

$$m_1 = w_{23} M_x - u_{23} M_z$$

$$m_2 = -(v_{24} M_x - u_{24} M_y)$$

$$m_3 = -(w_{24} M_x - u_{24} M_z)$$

$$m_4 = -(w_{24} M_y - v_{24} M_z)$$

$$m_5 = -(v_{25} M_x - u_{25} M_y)$$

$$m_6 = -(w_{25} M_x - u_{25} M_z)$$

$$m_7 = -(w_{25} M_y - v_{25} M_z)$$

$$n_1 = \tan \frac{\theta_{12}}{2}$$

$$n_2 = \tan \frac{\theta_{13}}{2}$$

$$n_3 = \tan \frac{\theta_{14}}{2}$$

$$n_4 = \tan \frac{\theta_{15}}{2}$$

In order to find a compatible solution for F_x , F_y and F_z , so that

all four Equations (3.13 - 3.16) will be satisfied simultaneously, let us consider first for $j = 2, 3, 4$, Equations (3.13 - 3.15); i. e.

$$\begin{vmatrix} a_1 & a_2 & 0 \\ a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 \end{vmatrix} = 0 \quad , \quad (3.17)$$

and then for $j = 2, 3, 5$, Equations (3.13), (3.14) and (3.16); i. e.

$$\begin{vmatrix} a_1 & a_2 & 0 \\ a_3 & a_4 & a_5 \\ a_9 & a_{10} & a_{11} \end{vmatrix} = 0 \quad . \quad (3.18)$$

Equations (3.17) and (3.18) when expanded result in the following cubic polynomials¹⁴:

$$\begin{aligned} T_{11} \ell^3 + T_{21} \ell^2 m + T_{31} \ell^2 n + T_{41} \ell m^2 + T_{51} \ell mn + \\ + T_{61} \ell n^2 + T_{71} m^3 + T_{81} m^2 n + T_{91} mn^2 = 0 \quad , \end{aligned} \quad (3.19)$$

and

$$\begin{aligned} T_{12} \ell^3 + T_{22} \ell^2 m + T_{32} \ell^2 n + T_{42} \ell m^2 + T_{52} \ell mn + \\ + T_{62} \ell n^2 + T_{72} m^3 + T_{82} m^2 n + T_{92} mn^2 = 0 \end{aligned} \quad (3.20)$$

where for simplicity $(\ell, m, n) = (M_x, M_y, M_z)$, and coefficients T_{i1} and T_{i2} , $i = 1, \dots, 9$ are defined in Table V.

An examination of Equations (3.19) and (3.20) shows that the coefficients of the n^3 term have vanished (these coefficients would not have vanished, had we not used the direction cosines in the transformed coordinate system). In this study, we desire to find the intersections of the two cubic cones shown above, and at the same time eliminate the undesired roots.

¹⁴ These two cubic polynomials are the equations of two cubic cones which are the space analogs to the planar circle-point curves.

TABLE V

COEFFICIENTS OF THE CUBIC CONES

$$T_{11} = w_{23}^2 [n_2 (v_{24} - c_4)] + u_{23} w_{23} [n_2 (c_3 - n_1 (w_{24} + c_6))] + w_{23} [n_1 (c_4 - v_{24})]$$

$$T_{21} = w_{23}^2 [n_2 (n_1 (c_4 - v_{24}) - (c_5 + u_{24}))] + u_{23} w_{23} [n_2 c_{13}] + u_{23} [c_3 - n_1 (c_6 + w_{24})] + w_{23} [n_1 (u_{24} + c_5)] + [n_1 n_2 (v_{24} - c_4)]$$

$$T_{31} = u_{23}^2 [n_2 (n_1 (c_6 + w_{24}) - c_3)] + w_{23}^2 [n_2 c_1] + u_{23} w_{23} [n_2 (n_1 (u_{24} - c_5) - 2 v_{24})] + u_{23} [n_1 (v_{24} - c_4)] + w_{23} [-n_1 c_1]$$

$$T_{41} = w_{23}^2 [n_1 n_2 (u_{24} + c_5)] + u_{23} w_{23} [n_2 (n_1 (c_6 - w_{24}) + c_2)] + u_{23} [c_{13}] + w_{23} [n_1 (c_4 - v_{24})] + [n_2 ((v_{24} - c_4) - n_1 (c_5 + u_{24}))]$$

$$T_{51} = u_{23}^2 [-n_2 c_{13}] + w_{23}^2 [-n_1 n_2 c_1] + u_{23} w_{23} [2 n_2 c_{14}] + u_{23} [-2 n_3 w_{24} c_{14}] + [n_1 n_2 c_1]$$

$$T_{61} = u_{23}^2 [n_2 (n_3 w_{24} c_{14} + (v_{24} - n_1 u_{24}))] + u_{23} w_{23} [-n_2 c_1] + u_{23} [n_1 c_1]$$

$$T_{71} = u_{23} [n_1 (c_6 - w_{24}) + c_2] + w_{23} [n_1 (c_5 + u_{24})] + [-n_2 (u_{24} + c_5)]$$

$$T_{81} = u_{23}^2 [n_2 (n_1 (w_{24} - c_6) - c_2)] + u_{23} w_{23} [-n_1 n_2 (u_{24} + c_5)] + u_{23} [n_1 (v_{24} + c_4) - 2 c_5] + w_{23} [-n_1 c_1] + [n_2 c_1]$$

$$T_{91} = u_{23}^2 [n_2 (-n_1 (c_4 + v_{24}) + (c_5 - u_{24}))] + u_{23} w_{23} [n_1 n_2 c_1] + u_{23} [c_1]$$

and

$$c_1 = n_3 (1 - w_{24}^2)$$

$$c_2 = n_3 (1 - v_{24}^2)$$

$$c_3 = n_3 (1 - u_{24}^2)$$

$$c_4 = n_3 u_{24} w_{24}$$

$$c_5 = n_3 v_{24} w_{24}$$

$$c_6 = n_3 u_{24} v_{24}$$

$$c_7 = n_4 (1 - w_{25}^2)$$

$$c_8 = n_4 (1 - v_{25}^2)$$

$$T_{12} = w_{23}^2 [n_2 (v_{25} - c_{10})] + u_{23} w_{23} [n_2 (c_9 - n_1 (w_{25} + c_{12}))] + w_{23} [n_1 (c_{10} - v_{25})]$$

$$T_{22} = w_{23}^2 [n_2 (n_1 (c_{10} - v_{25}) - (c_{11} + u_{25}))] + u_{23} w_{23} [n_2 c_{15}] + u_{23} [-n_1 (c_{12} + w_{25}) + c_9] + w_{23} [n_1 (u_{25} + c_{11})] + [n_1 n_2 (v_{25} - c_{10})]$$

$$T_{32} = u_{23}^2 [n_2 (n_1 (c_{12} + w_{24}) - c_9)] + w_{23}^2 [n_2 c_7] + u_{23} w_{23} [n_2 (n_1 (u_{25} - c_{11}) - 2 v_{25})] + u_{23} [n_1 (v_{25} - c_{10})] + w_{23} [-n_1 c_7]$$

$$T_{42} = w_{23}^2 [n_1 n_2 (u_{25} + c_{11})] + u_{23} w_{23} [n_2 (n_1 (c_{12} - w_{25}) + c_8)] + u_{23} [c_{15}] + w_{23} [n_1 (c_{10} - v_{25})] + [n_2 ((v_{25} - c_{10}) - n_1 (c_{11} + u_{25}))]$$

$$T_{52} = u_{23}^2 [-n_2 c_{15}] + w_{23}^2 [-n_1 n_2 c_7] + u_{23} w_{23} [2 n_2 c_{16}] + u_{23} [-2 n_4 w_{25} c_{16}] - [n_1 n_2 c_7]$$

$$T_{62} = u_{23}^2 [n_2 (n_4 w_{25} c_{16} + (v_{25} - n_1 u_{25}))] + u_{23} w_{23} [-n_2 c_7] + u_{23} [n_1 c_7]$$

$$T_{72} = u_{23} [n_1 (c_{12} - w_{25}) + c_8] + w_{23} [n_1 (c_{11} + u_{25})] + [-n_2 (c_{11} + u_{25})]$$

$$T_{82} = u_{23}^2 [n_2 (n_1 (w_{25} - c_{12}) - c_8)] + u_{23} w_{23} [-n_1 n_2 (u_{25} + c_{11})] + u_{23} [n_1 (v_{25} + c_{10}) - 2 c_{11}] + w_{23} [-n_1 c_7] + [n_2 c_7]$$

$$T_{92} = u_{23}^2 [n_2 ((c_{11} - u_{25}) - n_1 (c_{10} + v_{25}))] + u_{23} w_{23} [n_1 n_2 c_7] + u_{23} [c_9]$$

$$c_9 = n_4 (1 - u_{25}^2)$$

$$c_{10} = n_4 u_{25} w_{25}$$

$$c_{11} = n_4 v_{25} w_{25}$$

$$c_{12} = n_4 u_{25} v_{25}$$

$$c_{13} = n_3 [n_1 (u_{24}^2 - v_{24}^2) - 2 u_{24} v_{24}]$$

$$c_{14} = u_{24} + n_1 v_{24}$$

$$c_{15} = n_4 [n_1 (u_{25}^2 - v_{25}^2) - 2 u_{25} v_{25}]$$

$$c_{16} = u_{25} + n_1 v_{25}$$

In our case, we observe that Equations (3.17) and (3.18) both contain $j = 2, 3$; in other words both cubic cones represented by Equations (3.19) and (3.20) contain $\$_{12}$ and $\$_{13}$. Since these two screws are completely described in the transformed coordinate system (see Section 3.2), we can always find a third screw¹⁵ $\$_{23}'$ (also contained in both cubic cones) with direction cosines [105]

$$\left(-u_{23} \tan \frac{\theta_{13}}{2}, u_{23} \tan \frac{\theta_{12}}{2} \tan \frac{\theta_{13}}{2}, \tan \frac{\theta_{12}}{2} - w_{23} \tan \frac{\theta_{13}}{2}\right) \quad (3.2)$$

which is the screw, in the first position, fixed in the moving frame of reference which will coincide with $\$_{23}$ when the moving frame of reference is in position 2 or 3.

The three screws $\$_{12}$, $\$_{13}$, and $\$_{23}'$ are the undesired roots which we must eliminate from the intersection of the two cubic cones. At this point, we can simplify Equations (3.19) and (3.20) to eliminate the terms in n and n^2 . For this, we first need to rewrite these equations in the following form:

$$\begin{aligned} (w_{23}^{\ell} - u_{23}n)^2 (S_1^{\ell} + S_2^m) + (w_{23}^{\ell} - u_{23}n) (S_3^{\ell^2} + S_4^{\ell m} + \\ + S_5^{m^2}) + S_6^{\ell^2 m} + S_7^{\ell m^2} + S_8^{m^3} = 0 \end{aligned} \quad (3.22)$$

$$\begin{aligned} (w_{23}^{\ell} - u_{23}n)^2 (S_9^{\ell} + S_{10}^m) + (w_{23}^{\ell} - u_{23}n) (S_{11}^{\ell^2} + S_{12}^{\ell m} + \\ + S_{13}^{m^2}) + S_{14}^{\ell^2 m} + S_{15}^{\ell m^2} + S_{16}^{m^3} = 0 \end{aligned} \quad (3.23)$$

where S_i , $i = 1, \dots, 16$ are defined in Table VI.

Now, to eliminate n from equations (3.22) and (3.23), let us define

$$T = w_{23}^{\ell} - u_{23}n$$

$$R_1 = S_1^{\ell} + S_2^m$$

¹⁵ The geometrical construction for determining a single screw equivalent to two successive screws is known as Halphen's theorem [62].

TABLE VI

DEFINITION OF S_i , $i = 1, \dots, 16$

$$\begin{aligned}
S_1 &= k_1 n_1 + k_2 \\
S_2 &= -k_2 n_1 + k_1 \\
S_3 &= k_3 n_1 + k_4 \\
S_4 &= (k_8 - k_4) n_1 + k_6 \\
S_5 &= k_7 n_1 + k_8 \\
S_6 &= k_9 n_1 + k_{10} \\
S_7 &= (k_{14} - k_{10}) n_1 + (k_9 - k_{13}) \\
S_8 &= k_{13} n_1 + k_{14} \\
S_9 &= k_{15} n_1 + k_{16} \\
S_{10} &= -k_{16} n_1 + k_{15} \\
S_{11} &= k_{17} n_1 + k_{18} \\
S_{12} &= (k_{22} - k_{18}) n_1 + k_{20} \\
S_{13} &= k_{21} n_1 + k_{22} \\
S_{14} &= k_{23} n_1 + k_{24} \\
S_{15} &= (k_{28} - k_{24}) n_1 + (k_{23} - k_{27}) \\
S_{16} &= k_{27} n_1 + k_{28}
\end{aligned}$$

where

$$\begin{aligned}
k_1 &= n_2 (e_6 + e_8) + e_1 \\
k_2 &= n_2 e_2 \\
k_3 &= n_2 e_3 + e_4 \\
k_4 &= n_2 e_5 \\
k_5 &= k_8 - k_4 \\
k_6 &= n_2 (e_3 - e_7) + 2 (e_4 + v_{24}) \\
k_7 &= n_2 e_7 - e_2 \\
k_8 &= n_2 (e_9 - e_1) + e_6 \\
k_9 &= e_3 - n_2 e_4 \\
k_{10} &= e_5 \\
k_{11} &= k_{14} - k_{10} \\
k_{12} &= k_9 - k_{13} \\
k_{13} &= e_7 \\
k_{14} &= n_2 e_8 + e_9 \\
k_{15} &= n_2 (e_{15} + e_{17}) + e_{10} \\
k_{16} &= n_2 e_{11} \\
k_{17} &= n_2 e_{12} + e_{13} \\
k_{18} &= n_2 e_{14} \\
k_{19} &= k_{22} - k_{18} \\
k_{20} &= n_2 (e_{12} - e_{16}) + 2 (e_{13} + v_{25}) \\
k_{21} &= n_2 e_{16} - e_{11} \\
k_{22} &= n_2 (e_{18} - e_{10}) + e_{15} \\
k_{23} &= e_{12} - n_2 e_{13} \\
k_{24} &= e_{14} \\
k_{25} &= k_{28} - k_{24} \\
k_{26} &= k_{23} - k_{27} \\
k_{27} &= e_{16} \\
k_{28} &= n_2 e_{17} + e_{18}
\end{aligned}$$

and

$$\begin{aligned}
e_1 &= c_1/u_{23} \\
e_2 &= c_4 + v_{24} - c_1 w_{23}/u_{23} \\
e_3 &= w_{23}(u_{24} - c_5) - u_{23}(c_6 + w_{24}) \\
e_4 &= c_4 - v_{24} - c_1 w_{23}/u_{23} \\
e_5 &= u_{23} c_3 - 2c_4 w_{23} + c_1 w_{23}^2/u_{23} \\
e_6 &= 2c_5 \\
e_7 &= u_{23}(c_6 - w_{24}) + w_{23}(c_5 + u_{24}) \\
e_8 &= -(c_5 + u_{24}) \\
e_9 &= u_{23} c_2 \\
e_{10} &= c_7/u_{23} \\
e_{11} &= c_{10} + v_{25} - c_7 w_{23}/u_{23} \\
e_{12} &= w_{23}(u_{25} - c_{11}) - u_{23}(c_{12} + w_{25}) \\
e_{13} &= c_{10} - v_{25} - c_7 w_{23}/u_{23} \\
e_{14} &= u_{23} c_9 - 2c_{10} w_{23} + c_7 w_{23}^2/u_{23} \\
e_{15} &= 2c_{11} \\
e_{16} &= u_{23}(c_{12} - w_{25}) + w_{23}(c_{11} + u_{25}) \\
e_{17} &= -(c_{11} + u_{25}) \\
e_{18} &= u_{23} c_8
\end{aligned}$$

$$\begin{aligned}
R_2 &= S_3 \ell^2 + S_4 \ell m + S_5 m^2 \\
R_3 &= S_6 \ell^2 m + S_7 \ell m^2 + S_8 m^3 \\
R_4 &= S_9 \ell + S_{10} m \\
R_5 &= S_{11} \ell^2 + S_{12} \ell m + S_{13} m^2 \\
R_6 &= S_{14} \ell^2 m + S_{15} \ell m^2 + S_{16} m^3 .
\end{aligned}$$

This will permit us to rewrite these equations in the following form:

$$T^2 R_1 + T R_2 + R_3 = 0 \quad (3.24)$$

$$T^2 R_4 + T R_5 + R_6 = 0 \quad (3.25)$$

From the above, we can eliminate T by applying Sylvester's technique, thus

$$\begin{vmatrix}
R_1 & R_2 & R_3 & 0 \\
0 & R_1 & R_2 & R_3 \\
R_4 & R_5 & R_6 & 0 \\
0 & R_4 & R_5 & R_6
\end{vmatrix} = 0 \quad (3.26)$$

Expanding this determinant, we obtain a polynomial of degree seven, which is independent of any n -components, and at the same time does not contain the roots corresponding to $\$_{12}$ and $\$_{13}$ (this is true, since T is implicitly a function of $\$_{12}$ and $\$_{13}$). This seventh degree polynomial is of the form

$$b_0 p^7 + b_1 p^6 + b_2 p^5 + b_3 p^4 + b_4 p^3 + b_5 p^2 + b_6 p + b_7 = 0$$

where (3.27)

$$\begin{aligned}
b_0 &= f(n_1^4, n_1^3, n_1^2, n_1) \\
b_i, i &= 1, \dots, 6 = f(n_1^4, n_1^3, n_1^2, n_1, n_1^0) \\
b_7 &= f(n_1^3, n_1^2, n_1, n_1^0) \\
p &= \ell / m
\end{aligned}$$

Equation (3.21) gives the direction cosines (ℓ, m, n) of $\$_{23}$ '. By dividing ℓ/m in this equation, we obtain $p = -1/n_1$. By definition,

$p = \ell/m$, therefore, $(p n_1 + 1) = 0$ must be a root of Equation (3.27).

Since we know each of the coefficients b_i , $i = 1, \dots, 7$ of Equation (3.27) and the root $(p n_1 + 1)$ corresponding to $\$_{23}'$, we can eliminate this root from the seventh degree polynomial by synthetic division. This is not a simple task, since the coefficients of Equation (3.27) are very large, and the synthetic division must be carried out explicitly. The following steps outline the procedure used in this investigation:

1) We start by writing Equation (3.27) in the following form

$$(p n_1 + 1) (g_0 p^6 + g_1 p^5 + g_2 p^4 + g_3 p^3 + g_4 p^2 + g_5 p + g_6) = 0 \quad (3.28)$$

2) Then the coefficients corresponding to each of the powers of p for both Equations (3.27) and (3.28) are:

$$\begin{aligned} p^7: & \quad g_0 n_1 = b_0 \\ p^6: & \quad g_1 n_1 + g_0 = b_1 \\ p^5: & \quad g_2 n_1 + g_1 = b_2 \\ p^4: & \quad g_3 n_1 + g_2 = b_3 \\ p^3: & \quad g_4 n_1 + g_3 = b_4 \\ p^2: & \quad g_5 n_1 + g_4 = b_5 \\ p^1: & \quad g_6 n_1 + g_5 = b_6 \\ p^0: & \quad g_6 = b_7 \end{aligned}$$

3) To determine g_0 , let $b_0 = g_0 n_1 + R$. From this relationship, g_0 and R can be found¹⁶; here R is zero, the n_1^4 term is also zero, and $g_0 = f(n_1^3, n_1^2, n_1, n_1^0)$.

4) To find g_i , $i = 1, \dots, 6$, we have that $b_i - g_{i-1} = g_i n_1 + R$, $i = 1, \dots, 6$. Here again we find that the n_1^4 terms are zero, and $g_i = f(n_1^3, n_1^2, n_1, n_1^0)$. $(p n_1 + 1)$ is indeed a root of Equation (3.27) since g_6

¹⁶ This was accomplished with the computational system ALFRED [108].

is equal to b_7 .

The resulting polynomial, independent of screws, is of the form

$$g_0 p^6 + g_1 p^5 + g_2 p^4 + g_3 p^3 + g_4 p^2 + g_5 p + g_6 = 0 \quad (3.29)$$

where the coefficients g_i , $i = 0, \dots, 6$ are defined in Table VII.

The real roots of this sixth degree polynomial (3.29), give the ratios ℓ/m of the direction cosines (ℓ, m, n) of the Burmester lines in the moving rigid body. In order to completely describe these direction cosines, we first divide Equation (3.19) by m^3 , define ℓ/m as p , and n/m as q . Then, since all direction cosines must satisfy the condition $\ell^2 + m^2 + n^2 = 1$, we find that

$$m = 1 / \sqrt{p^2 + q^2 + 1},$$

and

$$\ell = p m$$

$$n = q m$$

The following reasoning was used to check if equation (3.29) is truly correct (see also Appendix E):

- 1) We checked Roth's [62] results with his own cubic polynomials. We found that the direction cosines of the moving Burmester lines given in his paper, when substituted in one of the cubic polynomials, force this polynomial to be equal to zero (to 5 significant digits). Since the direction cosines of the Burmester lines must force the cubic polynomials to be equal to zero, we conclude that his results are correct.
- 2) We transformed the Burmester lines given by Roth [62] into the new coordinate system derived in Section 3.2; then we substituted the transformed direction cosines corresponding to the Burmester lines into Equations (3.19) and (3.20), and we found that they are equal to zero (to 5 significant digits). Here again, since the direction

cosines of the Burmester lines must force the transformed cubic polynomial to be equal to zero, we conclude that equations (3.19) and (3.20) are correct.

- 3) Next, we defined R_i , $i = 1, \dots, 6$ by rewriting the cubic polynomials (3.19) and (3.20) in the form given by Equations (3.24) and (3.25). The 4×4 determinant given by Equation (3.26) must reduce to zero if the R_i are correct. By substituting the transformed direction cosines corresponding to the Burmester lines in the R_i , we find that the 4×4 determinant given by Equation (3.26) is indeed equal to zero (to 5 significant digits). Therefore, we can conclude that equation (3.26) is correct.
- 4) Expansion of Equation (3.26) resulted in a seventh degree polynomial given in Equation (3.27). By using synthetic division we eliminated root $(pn_1 + 1)$ from Equation (3.27) and obtained the sixth degree polynomial given in Equation (3.29). Since the residue after synthetic division is zero, we conclude that $(pn_1 + 1)$ is indeed a root of Equation (3.27).
- 5) Using Roth's [62] example, we found that the ratios ℓ/m found from the sixth degree polynomial (3.29) are in disagreement with the ratios ℓ/m corresponding to the transformed Burmester lines given by Roth. This disagreement can be justified by considering the amount of truncation and round-off errors that appear in the evaluation of each of the coefficients of the sixth degree polynomial (3.29). Nevertheless, the number of Burmester lines found with our procedure agrees exactly with Roth's.

The computational expenses to obtain Equation (3.29) were extremely high. There is still one more level of substitution, namely substituting

the k's by the e's as defined in Table VI. This portion of the research is left for the future generation of computers. Once the k's are substituted by the e's, we expect $(n_2^2 + 1)$ to factor out of Equation (3.29), thus simplifying the coefficients of the polynomial even more. Once this stage is reached, there is no limit on the research opportunities that Burmester theory in spatial kinematics can provide.

The procedure to determine the direction cosines of the Burmester lines in the fixed frame of reference, as well as the procedures to determine the locations of the Burmester lines in the fixed and the moving frames of reference are already available in the literature. These procedures are included in Appendices B, C and D for the sake of completeness.

TABLE VII
COEFFICIENTS OF THE SIXTH DEGREE POLYNOMIAL

$$\begin{aligned}
 g_0 &= T_{03} n_1^3 + T_{02} n_1^2 + T_{01} n_1 + T_{00} \\
 g_1 &= T_{13} n_1^3 + T_{12} n_1^2 + T_{11} n_1 + T_{10} \\
 g_2 &= T_{23} n_1^3 + T_{22} n_1^2 + T_{21} n_1 + T_{20} \\
 g_3 &= T_{33} n_1^3 + T_{32} n_1^2 + T_{31} n_1 + T_{30} \\
 g_4 &= T_{43} n_1^3 + T_{42} n_1^2 + T_{41} n_1 + T_{40} \\
 g_5 &= T_{53} n_1^3 + T_{52} n_1^2 + T_{51} n_1 + T_{50} \\
 g_6 &= T_{63} n_1^3 + T_{62} n_1^2 + T_{61} n_1 + T_{60}
 \end{aligned}$$

and T_{ij} , $i = 0, \dots, 6$, $j = 0, \dots, 3$ are¹⁷

¹⁷ In the following, * stands for multiplication and ** stands for exponentiation.

TABLE VII (continued)

$$T03 = -K1*K3*K17*K23+K1*K9*K17**2-K3*K9*K15*K17+K15*K23*K3**2$$

$$T02 = -K1*K3*K17*K24-K1*K3*K18*K23-K1*K4*K17*K23+2*K1*K9*K17*K18+K1*K10*K17**2-K2*K3*K17*K23+K2*K9*K17**2+2*K3*K4*K15*K23-(3*K9*K15*K18-K3*K9*K16*K17-K3*K10*K15*K17-K4*K9*K15*K17+K15*K24*K3**2+K16*K23*K3**2$$

$$T01 = -K1*K3*K18*K24-K1*K4*K17*K24-K1*K4*K18*K23+K1*K9*K18**2+2*K1*K10*K17*K18-K2*K3*K17*K24-K2*K3*K18*K23-K2*K4*K17*K23+2*K2*K9*K17*K18+K2*K10*K17**2+2*K3*K4*K15*K24+2*K3*K4*K16*K23-K3*K9*K16*K18-K3*K10*K15*K18-K3*K10*K16*K17-K4*K9*K15*K18-K4*K9*K16*K17-K4*K10*K15*K17+K15*K23*K4**2+K16*K24*K3**2$$

$$T00 = -K1*K4*K18*K24+K1*K10*K18**2-K2*K3*K18*K24-K2*K4*K17*K24-K2*K4*K18*K23+K2*K9*K18**2+2*K2*K10*K17*K18+2*K3*K4*K16*K24-K3*K10*K16*K18-K4*K9*K16*K18-K4*K10*K15*K18-K4*K10*K16*K17+K15*K24*K4**2+K16*K23*K4**2$$

$$T13 = K1*K3*K17*K24-K1*K3*K17*K28+K1*K3*K18*K23-K1*K3*K22*K23+K1*K4*K17*K23-K1*K8*K17*K23-2*K1*K9*K15*K23-2*K1*K9*K17*K18+2*K1*K9*K17*K22-K1*K10*K17**2+K1*K14*K17**2+K2*K3*K17*K23-K2*K9*K17**2-2*K3*K4*K15*K23+2*K3*K8*K15*K23+K3*K9*K15*K18-K3*K9*K15*K22+K3*K9*K16*K17+K3*K10*K15*K17-K3*K14*K15*K17+K4*K9*K15*K17-K4*K9*K15*K17-K15*K24*K3**2+K15*K28*K3**2-K16*K23*K3**2+K1**2*K23**2+K9**2*K15**2$$

$$T12 = 2*K1*K2*K23**2-K1*K3*K17*K23+K1*K3*K17*K27+2*K1*K3*K18*K24-K1*K3*K18*K28-K1*K3*K20*K23-K1*K3*K22*K24+2*K1*K4*K17*K24-K1*K4*K17*K28+2*K1*K4*K18*K23-K1*K4*K22*K23-K1*K6*K17*K23-K1*K8*K17*K24-K1*K8*K18*K23-2*K1*K9*K15*K24-2*K1*K9*K16*K23+2*K1*K9*K17*K20+2*K1*K9*K18*K22+K1*K9*K17**2-2*K1*K9*K18**2-2*K1*K10*K15*K23-4*K1*K10*K17*K18+2*K1*K10*K17*K22-K1*K13*K17**2+2*K1*K14*K17*K18+2*K2*K3*K17*K24-K2*K3*K17*K28+2*K2*K3*K18*K23-K2*K3*K22*K23+2*K2*K4*K17*K23-K2*K8*K17*K23-2*K2*K9*K15*K23-4*K2*K9*K17*K18+2*K2*K9*K17*K22-2*K2*K10*K17**2+2*K2*K14*K17**2-4*K3*K4*K15*K24+2*K3*K4*K15*K28-4*K3*K4*K16*K23+2*K3*K6*K15*K23+2*K3*K8*K15*K24+2*K3*K8*K16*K23-K3*K9*K15*K17-K3*K9*K15*K20+2*K3*K9*K16*K18-K3*K9*K16*K22+2*K3*K10*K15*K18-K3*K10*K15*K22+2*K3*K10*K16*K17+K3*K13*K15*K17-K3*K14*K15*K18-K3*K14*K16*K17+2*K4*K8*K15*K23+2*K4*K9*K15*K18-K4*K9*K15*K22+2*K4*K9*K16*K17+2*K4*K10*K15*K17-K4*K14*K15*K17-K6*K9*K15*K17-K8*K9*K15*K18-K8*K9*K16*K17-K8*K10*K15*K17+2*K9*K10*K15**2+2*K15*K16*K9**2+K15*K23*K3**2-2*K15*K23*K4**2-K15*K27*K3**2-2*K16*K24*K3**2+K16*K28*K3**2+2*K23*K24*K1**2$$

$$T11 = 4*K1*K2*K23*K24-K1*K3*K18*K23+K1*K3*K18*K27-K1*K3*K20*K24-K1*K4*K17*K23+K1*K4*K17*K27+3*K1*K4*K18*K24-K1*K4*K18*K28-K1*K4*K20*K23-K1*K4*K22*K24-K1*K6*K17*K24-K1*K6*K18*K23-K1*K8*K18*K24-2*K1*K9*K16*K24+2*K1*K9*K17*K18+2*K1*K9*K18*K20-2*K1*K10*K15*K24-2*K1*K10*K16*K23+2*K1*K10*K17*K20+2*K1*K10*K18*K22-3*K1*K10*K18**2-2*K1*K13*K17*K18+K1*K14*K18**2+K2*K3*K17*K27+3*K2*K3*K18*K24-K2*K3*K18*K28-K2*K3*K20*K23-K2*K3*K22*K24+3*K2*K4*K17*K24-K2*K4*K17*K28+3*K2*K4*K18*K23-2*K4*K22*K23-K2*K6*K17*K23-K2*K8*K17*K24-K2*K8*K18*K23-2*K2*K9*K15*K24-2*K2*K9*K16*K23+2*K2*K9*K17*K20+2*K2*K9*K18*K22-3*K2*K9*K18**2-2*K2*K10*K15*K23-6*K2*K10*K17*K18+2*K2*K10*K17*K22-K2*K13*K17**2+2*K2*K14*K17*K18+2*K3*K4*K15*K23-2*K3*K4*K15*K27-6*K3*K4*K16*K24+2*K3*K4*K16*K28+2*K3*K6*K15*K24+2*K3*K6*K16*K23+2*K3*K8*K16*K24-K3*K9*K15*K18-K3*K9*K16*K20-K3*K10*K15*K20+3*K3*K10*K16*K18-K3*K10*K16*K22+K3*K13*K15*K18+K3*K13*K16*K17-K3*K14*K16*K18+2*K4*K6*K15*K23+2*K4*K8*K15*K24+2*K4*K8*K16*K23-K4*K9*K15*K17-K4*K9*K15*K20+3*K4*K9*K16*K18-K4*K9*K16*K22+3*K4*K10*K15*K18-K4*K10*K15*K17-K6*K9*K15*K18-K6*K9*K16*K17-K6*K10*K15*K17-K8*K9*K16*K18-K8*K10*K15*K18-K8*K10*K16*K17+4*K9*K10*K15*K16-3*K15*K24*K4**2+K15*K28*K4**2-3*K16*K23*K4**2-K16*K27*K3**2+K1**2*K24**2+K2**2*K23**2+K9**2*K15**2+K10**2*K15**2$$

TABLE VII (continued)

T10 = 2*K1*K2*K24**2-K1*K4*K18*K23+K1*K4*K13*K27-K1*K4*K20*K24-K1*K6*K13*K24+K1*K9*K18**2-2*K1*K10*K16*K24+2*K1*K10*K18*K20-K1*K13*K18**2+K2*K3*K17*K24+K2*K3*K18*K27-K2*K3*K20*K24+K2*K4*K17*K27+4*K2*K4*K18*K24-K2*K4*K18*K28-K2*K4*K20*K23-K2*K4*K22*K24-K2*K6*K17*K24-K2*K6*K18*K23-K2*K8*K18*K24-2*K2*K9*K16*K24+2*K2*K9*K18*K20-2*K2*K10*K15*K24-2*K2*K10*K16*K23+2*K2*K10*K17*K20+2*K2*K10*K18*K22-K2*K10*K17**2-4*K2*K10*K18**2-2*K2*K13*K17*K18+K2*K14*K18**2-2*K3*K4*K16*K27+2*K3*K6*K16*K24+K3*K10*K16*K17-K3*K10*K16*K20+K3*K13*K16*K18+2*K4*K6*K15*K24+2*K4*K6*K16*K23+2*K4*K6*K16*K24-K4*K9*K15*K13-K4*K9*K16*K20-K4*K10*K15*K20+4*K4*K10*K16*K18-K4*K10*K16*K22+K4*K13*K15*K18+K4*K13*K16*K17-K4*K14*K16*K18-K6*K9*K16*K18-K6*K10*K15*K18-K6*K10*K16*K17-K8*K10*K16*K18+2*K9*K10*K16**2+2*K15*K10*K10**2+K15*K23*K4**2-K15*K27*K4**2-K16*K24*K3**2-4*K16*K24*K4**2+K16*K28*K4**2+2*K23*K24*K2**2

T23 = -2*K1*K2*K23**2-K1*K3*K17*K27-K1*K3*K18*K24+K1*K3*K18*K28-K1*K3*K21*K23+K1*K3*K22*K24-K1*K3*K22*K28-K1*K4*K17*K24+K1*K4*K17*K28-K1*K4*K18*K23+K1*K4*K22*K23-K1*K7*K17*K23+K1*K8*K17*K24-K1*K8*K17*K28+K1*K8*K18*K23-K1*K8*K22*K23+2*K1*K9*K15*K24-2*K1*K9*K15*K28+2*K1*K9*K16*K23+2*K1*K9*K17*K21-2*K1*K9*K18*K22+K1*K9*K18**2+K1*K9*K22**2+2*K1*K10*K15*K23+4*K1*K10*K17*K18-4*K1*K10*K17*K22+K1*K13*K17**2-2*K1*K14*K15*K23-4*K1*K14*K17*K18+4*K1*K14*K17*K22-K2*K3*K17*K24+K2*K3*K17*K28-K2*K3*K18*K23+K2*K3*K22*K23-K2*K4*K17*K23+K2*K8*K17*K23+2*K2*K9*K15*K23+2*K2*K9*K17*K18-2*K2*K9*K17*K22+K2*K10*K17**2-K2*K14*K17**2+2*K3*K4*K15*K24-2*K3*K4*K15*K28+2*K3*K4*K16*K23+2*K3*K7*K15*K23-2*K3*K8*K15*K24+2*K3*K8*K15*K28-2*K3*K8*K16*K23-K3*K9*K15*K21-K3*K9*K16*K18+K3*K9*K16*K22-K3*K10*K15*K18+K3*K10*K15*K22-K3*K10*K16*K17-K3*K13*K15*K17+K3*K14*K15*K15*K18-K3*K14*K15*K22+K3*K14*K16*K17-2*K4*K8*K15*K23-K4*K9*K15*K18+K4*K9*K15*K22-K4*K9*K16*K17-K4*K10*K15*K17+K4*K14*K15*K17-K7*K9*K15*K17+K8*K9*K15*K18-K8*K9*K15*K22+K8*K9*K16*K17+K8*K10*K15*K17-K8*K14*K15*K17-2*K9*K10*K15**2+2*K9*K14*K15**2-2*K15*K16*K9**2+K15*K23*K4**2+K15*K23*K8**2+K15*K27*K3**2+K16*K24*K3**2-K16*K28*K3**2-2*K23*K24*K1**2+2*K23*K28*K1**2

T22 = -8*K1*K2*K23*K24+4*K1*K2*K23*K28-K1*K3*K17*K28+K1*K3*K18*K23-2*K1*K3*K18*K27+K1*K3*K20*K24-K1*K3*K20*K28-K1*K3*K21*K24-2*K1*K3*K22*K23+K1*K3*K22*K27+K1*K4*K17*K23-2*K1*K4*K17*K27-3*K1*K4*K18*K24+2*K1*K4*K18*K28+K1*K4*K20*K23-K1*K4*K21*K23+2*K1*K4*K22*K24-K1*K4*K22*K28+K1*K6*K17*K24-K1*K6*K17*K28+K1*K6*K18*K23-K1*K6*K22*K23-K1*K7*K17*K24-K1*K7*K18*K23-2*K1*K8*K17*K23+K1*K8*K17*K27+2*K1*K8*K18*K24-K1*K8*K18*K28-K1*K8*K20*K23-K1*K8*K22*K24-6*K1*K9*K15*K23+2*K1*K9*K15*K27+4*K1*K9*K16*K24-2*K1*K9*K16*K28-2*K1*K9*K17*K18+4*K1*K9*K17*K22-2*K1*K9*K18*K20+2*K1*K9*K18*K21+2*K1*K9*K20*K22+4*K1*K10*K15*K24-2*K1*K10*K15*K28+4*K1*K10*K16*K23-4*K1*K10*K17*K20+2*K1*K10*K17*K21-6*K1*K10*K18*K22+5*K1*K10*K18**2+K1*K10*K22**2+2*K1*K13*K15*K23+4*K1*K13*K17**2-K1*K13*K17*K22-2*K1*K14*K15*K24-2*K1*K14*K16*K23+4*K1*K14*K17*K20+4*K1*K14*K18*K22+K1*K14*K17**2-4*K1*K14*K18**2-2*K2*K3*K17*K27-3*K2*K3*K18*K24+2*K2*K3*K18*K28+K2*K3*K20*K23-K2*K3*K21*K23+2*K2*K3*K22*K24-K2*K3*K22*K28-3*K2*K4*K17*K24+2*K2*K4*K17*K28-3*K2*K4*K18*K23+2*K2*K4*K22*K23+K2*K6*K17*K23-K2*K7*K17*K23+2*K2*K8*K17*K24-K2*K8*K17*K28+2*K2*K9*K18*K23-K2*K9*K22*K23+4*K2*K9*K15*K24-2*K2*K9*K15*K28+4*K2*K9*K16*K23-2*K2*K9*K17*K20+2*K2*K9*K17*K21-4*K2*K9*K18*K22+3*K2*K9*K18**2+K2*K9*K22**2+4*K2*K10*K15*K23+6*K2*K10*K17*K18-4*K2*K10*K17*K22+2*K2*K13*K17**2-2*K2*K14*K15*K23-4*K2*K14*K17*K18+2*K2*K14*K17*K22-2*K3*K4*K15*K23+4*K3*K4*K15*K27+6*K3*K4*K16*K24-4*K3*K4*K16*K28-2*K3*K6*K15*K24+2*K3*K6*K15*K28-2*K3*K6*K16*K23+2*K3*K7*K15*K24+2*K3*K7*K16*K23+4*K3*K8*K15*K23-2*K3*K8*K15*K27-4*K3*K8*K16*K24+2*K3*K8*K16*K28+K3*K9*K15*K18-2*K3*K9*K15*K22+K3*K9*K16*K20-K3*K9*K16*K21+K3*K10*K15*K20-K3*K10*K15*K21-3*K3*K10*K16*K18+2*K3*K10*K16*K22-2*K3*K13*K15*K18+K3*K13*K15*K22-2*K3*K13*K16*K17-K3*K14*K15*K17-K3*K14*K15*K20+2*K3*K14*K16*K18-K3*K14*K16*K22-2*K4*K6*K15*K23+2*K4*K7*K15*K23-4*K4*K8*K15*K24+2*K4*K8*K15*K26-4*K4*K8*K16*K23+4*K4*K9*K15*K17+K4*K9*K15*K20-K4*K9*K15*K21-3*K4*K9*K16*K16*K18+2*K4*K9*K16*K22-3*K4*K10*K15*K18+2*K4*K10*K15*K22-3*K4*K10*K16*K17-2*K4*K13*K15*K17+2*K4*K14*K15*K18-K4*K14*K15*K22+2*K4*K14*K16*K17+2*K6*K10*K15*K23+K6*K9*K15*K18-K6*K9*K15*K22+K6*K9*K16*K17+K6*K10*K15*K17-K6*K14*K15*K17-K7*K9*K15*K18-K7*K9*K16*K17-K7*K10*K15*K17-2*K8*K9*K15*K17-K8*K9*K15*K20+2*K8*K9*K16*K18-K8*K9*K16*K22+2*K8*K10*K15*K16-K8*K10*K15*K22+2*K8*K10*K16*K17+K8*K13*K15*K17-K8*K14*K15*K18-K8*K14*K16*K17-8*K9*K10*K15*K16-2*K9*K13*K15**2+4*K9*K14*K15*K16+2*K10*K14*K15**2+3*K15*K24*K4**2+K15*K24*K8**2+K15*K28*K3**2-2*K15*K28*K4**2+3*K16*K23*K4**2+K15*K23*K8**2+2*K16*K27*K3**2-2*K23*K27*K1**2+2*K24*K28*K1**2+3*K1**2*K23**2-2*K1**2*K24**2-2*K2**2*K23**2+3*K9**2*K15**2-2*K9**2*K16**2-2*K10**2*K15**2

TABLE VII (continued)

$$\begin{aligned}
T21 = & -4 * K1 * K2 * K23 * K27 + 4 * K1 * K2 * K24 * K28 + 4 * K1 * K2 * K23 * K27 - 6 * K1 * K2 * \\
& K24 * K2 - K1 * K3 * K18 * K28 - K1 * K3 * K20 * K23 + K1 * K3 * K20 * K27 - K1 * K3 * K22 * K24 - K1 * K4 * \\
& K17 * K28 + 2 * K1 * K4 * K18 * K23 - 3 * K1 * K4 * K18 * K27 + 2 * K1 * K4 * K20 * K24 - K1 * K4 * K20 * K28 - \\
& K1 * K4 * K21 * K24 - 2 * K1 * K4 * K22 * K23 + K1 * K4 * K22 * K27 - K1 * K6 * K17 * K23 + K1 * K6 * K17 * K27 \\
& + 2 * K1 * K6 * K18 * K24 - K1 * K6 * K18 * K28 - K1 * K6 * K20 * K23 - K1 * K6 * K22 * K24 - K1 * K7 * K18 * \\
& K24 - K1 * K8 * K17 * K24 - 2 * K1 * K8 * K18 * K23 + K1 * K8 * K18 * K27 - K1 * K8 * K20 * K24 - 4 * K1 * K9 * \\
& K15 * K24 - 4 * K1 * K9 * K16 * K23 + 2 * K1 * K9 * K16 * K27 + 2 * K1 * K9 * K17 * K20 + 4 * K1 * K9 * K18 * K22 \\
& - 2 * K1 * K9 * K18 * K27 + K1 * K9 * K20 * K22 - 4 * K1 * K10 * K15 * K23 + 2 * K1 * K10 * K15 * K27 + 4 * K1 * K10 * \\
& K16 * K24 - 2 * K1 * K10 * K16 * K28 - 2 * K1 * K10 * K17 * K18 + 2 * K1 * K10 * K17 * K22 - 4 * K1 * K10 * \\
& K18 * K20 + 2 * K1 * K10 * K18 * K21 + 2 * K1 * K10 * K20 * K22 + 2 * K1 * K13 * K15 * K24 + 2 * K1 * K13 * K16 * \\
& K23 - 2 * K1 * K13 * K17 * K20 - 2 * K1 * K13 * K18 * K22 + 3 * K1 * K13 * K18 * K27 + 4 * K1 * K14 * K17 * K18 \\
& + 2 * K1 * K14 * K18 * K20 - 2 * K2 * K3 * K17 * K24 - 3 * K2 * K3 * K18 * K27 + 2 * K2 * K3 * K20 * K24 - K2 * K3 * \\
& K20 * K28 - K2 * K3 * K21 * K24 - K2 * K3 * K22 * K23 + K2 * K3 * K22 * K27 - 3 * K2 * K4 * K17 * K27 - 6 * K2 * \\
& K4 * K18 * K24 + 3 * K2 * K4 * K18 * K28 + 2 * K2 * K4 * K20 * K23 - K2 * K4 * K21 * K23 + 3 * K2 * K4 * K22 * \\
& K24 - K2 * K4 * K22 * K28 + 2 * K2 * K6 * K17 * K24 - K2 * K6 * K17 * K28 + 2 * K2 * K6 * K18 * K23 - K2 * K6 * \\
& K22 * K23 - K2 * K7 * K17 * K24 - K2 * K7 * K18 * K23 - K2 * K8 * K17 * K23 + K2 * K8 * K17 * K27 + 3 * K2 * K8 * \\
& K18 * K24 - K2 * K8 * K18 * K28 - K2 * K8 * K20 * K23 - K2 * K8 * K22 * K24 - 4 * K2 * K9 * K15 * K23 + 2 * K2 * \\
& K9 * K15 * K27 + 6 * K2 * K9 * K16 * K24 - 2 * K2 * K9 * K16 * K28 + 2 * K2 * K9 * K17 * K22 - 4 * K2 * K9 * K18 * \\
& K20 + 2 * K2 * K9 * K18 * K21 + 2 * K2 * K9 * K20 * K22 + 6 * K2 * K10 * K15 * K24 - 2 * K2 * K10 * K15 * K28 + \\
& 6 * K2 * K10 * K16 * K23 - 4 * K2 * K10 * K17 * K20 + 2 * K2 * K10 * K17 * K21 - 6 * K2 * K10 * K18 * K22 + 2 * \\
& K2 * K10 * K17 * K27 + 6 * K2 * K10 * K18 * K27 + 2 * K2 * K10 * K22 * K27 + 2 * K2 * K13 * K15 * K23 + 6 * K2 * K13 * \\
& K17 * K18 - 2 * K2 * K13 * K17 * K22 - 2 * K2 * K14 * K15 * K24 - 2 * K2 * K14 * K16 * K23 + 2 * K2 * K14 * K17 * \\
& K20 + 2 * K2 * K14 * K18 * K22 - 3 * K2 * K14 * K18 * K27 + 2 * K3 * K4 * K15 * K28 + 6 * K3 * K4 * K16 * K27 + 2 * \\
& K3 * K6 * K15 * K23 - 2 * K3 * K6 * K15 * K27 - 4 * K3 * K6 * K16 * K24 + 2 * K3 * K6 * K16 * K28 + 2 * K3 * K7 * \\
& K16 * K24 + 2 * K3 * K8 * K15 * K24 + 2 * K3 * K8 * K16 * K23 - 2 * K3 * K8 * K16 * K27 - K3 * K9 * K15 * K20 - \\
& K3 * K9 * K16 * K22 - K3 * K10 * K15 * K22 - 2 * K3 * K10 * K16 * K17 + 2 * K3 * K10 * K16 * K20 - K3 * K10 * \\
& K16 * K21 + K3 * K13 * K15 * K20 - 3 * K3 * K13 * K16 * K18 + K3 * K13 * K16 * K22 - K3 * K14 * K15 * K18 - \\
& K3 * K14 * K16 * K20 - 4 * K4 * K6 * K15 * K24 + 2 * K4 * K6 * K15 * K28 - 4 * K4 * K6 * K16 * K23 + 2 * K4 * K7 * \\
& K15 * K24 + 2 * K4 * K7 * K16 * K23 + 4 * K4 * K8 * K15 * K23 - 2 * K4 * K8 * K15 * K27 - 6 * K4 * K8 * K16 * K24 \\
& + 2 * K4 * K8 * K16 * K28 + 2 * K4 * K9 * K15 * K18 - 2 * K4 * K9 * K15 * K22 + 2 * K4 * K9 * K16 * K20 - K4 * K9 * \\
& K16 * K21 + 2 * K4 * K10 * K15 * K20 - K4 * K10 * K15 * K21 - 6 * K4 * K10 * K16 * K18 + 3 * K4 * K10 * K16 * \\
& K22 - 3 * K4 * K13 * K15 * K18 + K4 * K13 * K15 * K22 - 3 * K4 * K13 * K16 * K17 - K4 * K14 * K15 * K17 - K4 * \\
& K14 * K15 * K20 + 3 * K4 * K14 * K16 * K18 - K4 * K14 * K16 * K22 + 2 * K6 * K8 * K15 * K24 + 2 * K6 * K8 * K16 * \\
& K23 - K6 * K9 * K15 * K17 - K6 * K9 * K15 * K20 + 2 * K6 * K9 * K16 * K18 - K6 * K9 * K16 * K22 + 2 * K6 * K10 * \\
& K15 * K18 - K6 * K10 * K15 * K22 + 2 * K6 * K10 * K16 * K17 + K6 * K13 * K15 * K17 - K6 * K14 * K15 * K18 - \\
& K6 * K14 * K16 * K17 - K7 * K9 * K16 * K18 - K7 * K10 * K15 * K18 - K7 * K10 * K16 * K17 - 2 * K8 * K9 * K15 * \\
& K18 - K8 * K9 * K16 * K17 - K8 * K9 * K16 * K20 - K8 * K10 * K15 * K17 - K8 * K10 * K15 * K20 + 3 * K8 * K10 * \\
& K16 * K18 - K8 * K10 * K16 * K22 + K8 * K13 * K15 * K18 + K8 * K13 * K16 * K17 - K8 * K14 * K16 * K18 + 4 * \\
& K9 * K10 * K15 * K2 - 6 * K9 * K10 * K16 * K2 - 4 * K9 * K13 * K15 * K16 + 2 * K9 * K14 * K16 * K2 - 2 * K10 * \\
& K13 * K15 * K2 + 4 * K10 * K14 * K15 * K16 + 4 * K15 * K16 * K9 * K2 - 6 * K15 * K16 * K10 * K2 - 2 * K15 * K23 \\
& K4 * K2 + K15 * K23 * K6 * K2 + 3 * K15 * K27 * K4 * K2 + 2 * K16 * K24 * K3 * K2 + 6 * K16 * K24 * K4 * K2 + \\
& K16 * K24 * K8 * K2 - 3 * K16 * K28 * K4 * K2 + 4 * K23 * K24 * K1 * K2 - 6 * K23 * K24 * K2 * K2 + 2 * K23 * K28 \\
& K2 * K2 - 2 * K24 * K27 * K1 * K2
\end{aligned}$$

TABLE VII (continued)

$T20 = 4 * K1 * K2 * K23 * K24 - 4 * K1 * K2 * K24 * K27 - K1 * K4 * K18 * K28 - K1 * K4 * K20 * K23 + K1 * K4 * K20 * K27 - K1 * K4 * K22 * K24 - K1 * K6 * K18 * K23 + K1 * K6 * K18 * K27 - K1 * K6 * K20 * K24 - K1 * K8 * K18 * K24 - 2 * K1 * K9 * K16 * K24 + 2 * K1 * K9 * K18 * K20 - 2 * K1 * K10 * K15 * K24 - 2 * K1 * K10 * K16 * K23 + 2 * K1 * K10 * K16 * K27 + 2 * K1 * K10 * K18 * K22 + K1 * K10 * K20 * K22 + 2 * K1 * K13 * K16 * K24 - 2 * K1 * K13 * K18 * K20 + K1 * K14 * K18 * K22 - K2 * K3 * K17 * K27 - 3 * K2 * K3 * K18 * K24 + K2 * K3 * K20 * K27 - 3 * K2 * K4 * K17 * K24 - 4 * K2 * K4 * K18 * K27 + 3 * K2 * K4 * K20 * K24 - K2 * K4 * K20 * K28 - K2 * K4 * K21 * K24 - K2 * K4 * K22 * K23 + K2 * K4 * K22 * K27 + K2 * K6 * K17 * K27 + 3 * K2 * K6 * K18 * K24 - K2 * K6 * K18 * K28 - K2 * K6 * K20 * K23 - K2 * K6 * K22 * K24 - K2 * K7 * K18 * K24 - K2 * K8 * K18 * K23 + K2 * K8 * K18 * K27 - K2 * K8 * K20 * K24 - 2 * K2 * K9 * K15 * K24 - 2 * K2 * K9 * K16 * K23 + 2 * K2 * K9 * K16 * K27 + 2 * K2 * K9 * K18 * K22 + K2 * K9 * K20 * K22 - 2 * K2 * K10 * K15 * K23 + 2 * K2 * K10 * K15 * K27 + 8 * K2 * K10 * K16 * K24 - 2 * K2 * K10 * K16 * K28 + 6 * K2 * K10 * K17 * K18 - 6 * K2 * K10 * K18 * K20 + 2 * K2 * K10 * K18 * K21 + 2 * K2 * K10 * K20 * K22 + 2 * K2 * K13 * K15 * K24 + 2 * K2 * K13 * K16 * K23 - 2 * K2 * K13 * K17 * K20 - 2 * K2 * K13 * K18 * K22 + K2 * K13 * K17 * K22 + 4 * K2 * K13 * K18 * K22 - 2 * K2 * K14 * K16 * K24 + 2 * K2 * K14 * K18 * K20 + 6 * K3 * K4 * K16 * K24 - 2 * K3 * K6 * K16 * K27 - 3 * K3 * K10 * K16 * K18 - K3 * K13 * K16 * K17 + K3 * K13 * K16 * K20 + 2 * K4 * K6 * K15 * K23 - 2 * K4 * K6 * K15 * K27 - 6 * K4 * K6 * K16 * K24 + 2 * K4 * K6 * K16 * K28 + 2 * K4 * K7 * K16 * K24 + 2 * K4 * K8 * K15 * K24 + 2 * K4 * K8 * K16 * K23 - 2 * K4 * K8 * K16 * K27 - K4 * K9 * K15 * K20 - K4 * K9 * K16 * K22 - K4 * K10 * K15 * K22 - 3 * K4 * K10 * K16 * K17 + 3 * K4 * K10 * K16 * K20 - K4 * K10 * K16 * K21 + K4 * K13 * K15 * K20 - 4 * K4 * K13 * K16 * K18 + K4 * K13 * K16 * K22 - K4 * K14 * K15 * K18 - K4 * K14 * K16 * K20 + 2 * K6 * K8 * K16 * K24 - K6 * K9 * K15 * K18 - K6 * K9 * K16 * K20 - K6 * K10 * K15 * K20 + 3 * K6 * K17 * K16 * K18 - K6 * K19 * K16 * K22 + K6 * K13 * K15 * K18 + K6 * K13 * K16 * K17 - K6 * K14 * K16 * K18 - K7 * K10 * K16 * K18 - K9 * K9 * K16 * K18 - K8 * K10 * K15 * K18 - K8 * K10 * K16 * K20 + K9 * K13 * K16 * K18 + 4 * K9 * K10 * K15 * K16 - 2 * K9 * K13 * K16 * K22 - 4 * K10 * K13 * K15 * K16 + 2 * K10 * K14 * K16 * K22 + K15 * K24 * K6 * K22 + K15 * K28 * K4 * K22 + K16 * K23 * K6 * K22 + K16 * K27 * K3 * K22 + 4 * K16 * K27 * K4 * K22 - 2 * K23 * K27 * K2 * K22 + 2 * K24 * K28 * K2 * K22 + K1 * K2 * K24 * K22 + K2 * K2 * K23 * K22 - 4 * K2 * K2 * K24 * K22 + 9 * K2 * K16 * K22 + K10 * K2 * K15 * K22 - 4 * K10 * K2 * K16 * K22$

$T33 = 4 * K1 * K2 * K23 * K24 - 4 * K1 * K2 * K23 * K28 + K1 * K3 * K18 * K27 + K1 * K3 * K21 * K24 - K1 * K3 * K21 * K28 - K1 * K3 * K22 * K27 + K1 * K4 * K17 * K27 + K1 * K4 * K18 * K24 - K1 * K4 * K18 * K28 + K1 * K7 * K18 * K23 - K1 * K7 * K22 * K23 - K1 * K8 * K17 * K27 - K1 * K8 * K18 * K24 + K1 * K8 * K18 * K28 - K1 * K8 * K21 * K23 + K1 * K8 * K22 * K24 - K1 * K8 * K22 * K28 - 2 * K1 * K9 * K15 * K27 - 2 * K1 * K9 * K16 * K24 + 2 * K1 * K9 * K16 * K28 - 2 * K1 * K9 * K18 * K21 + 2 * K1 * K9 * K21 * K22 - 2 * K1 * K10 * K15 * K24 + 2 * K1 * K10 * K15 * K28 - 2 * K1 * K10 * K16 * K23 - 2 * K1 * K10 * K17 * K21 + 2 * K1 * K10 * K18 * K22 - K1 * K10 * K18 * K22 - K1 * K10 * K22 * K22 - 2 * K1 * K13 * K15 * K23 - 2 * K1 * K13 * K17 * K18 + 2 * K1 * K13 * K17 * K22 + 2 * K1 * K14 * K15 * K24 - 2 * K1 * K14 * K15 * K28 + 2 * K1 * K14 * K16 * K23 + 2 * K1 * K14 * K17 * K21 - 2 * K1 * K14 * K18 * K22 + K1 * K14 * K18 * K22 + K1 * K14 * K22 * K22 + K2 * K3 * K17 * K27 + K2 * K3 * K18 * K24 - K2 * K3 * K18 * K28 + K2 * K3 * K21 * K23 - K2 * K3 * K22 * K24 + K2 * K3 * K22 * K28 + K2 * K4 * K17 * K24 - K2 * K4 * K17 * K28 + K2 * K4 * K18 * K23 - K2 * K4 * K22 * K23 + K2 * K7 * K17 * K23 - K2 * K8 * K17 * K24 + K2 * K8 * K17 * K28 - K2 * K8 * K18 * K23 + K2 * K8 * K22 * K23 - 2 * K2 * K9 * K15 * K24 + 2 * K2 * K9 * K15 * K28 - 2 * K2 * K9 * K16 * K23 - 2 * K2 * K9 * K17 * K21 + 2 * K2 * K9 * K18 * K22 - K2 * K9 * K18 * K22 - K2 * K9 * K22 * K22 - 2 * K2 * K10 * K15 * K23 - 2 * K2 * K10 * K17 * K18 + 2 * K2 * K10 * K17 * K22 - K2 * K13 * K17 * K22 + 2 * K2 * K14 * K15 * K23 + 2 * K2 * K14 * K17 * K18 - 2 * K2 * K14 * K17 * K22 - 2 * K3 * K4 * K15 * K27 - 2 * K3 * K4 * K16 * K24 + 2 * K3 * K4 * K16 * K28 - 2 * K3 * K7 * K15 * K24 + 2 * K3 * K7 * K15 * K28 - 2 * K3 * K7 * K16 * K23 + 2 * K3 * K8 * K15 * K27 + 2 * K3 * K8 * K16 * K24 - 2 * K3 * K8 * K16 * K28 + K3 * K9 * K16 * K21 + K3 * K10 * K15 * K21 + K3 * K10 * K16 * K18 - K3 * K10 * K16 * K22 + K3 * K13 * K15 * K18 - K3 * K13 * K15 * K22 + K3 * K13 * K16 * K17 - K3 * K14 * K15 * K21 - K3 * K14 * K16 * K18 + K3 * K14 * K16 * K22 - 2 * K4 * K7 * K15 * K23 + 2 * K4 * K8 * K15 * K24 - 2 * K4 * K8 * K15 * K28 + 2 * K4 * K8 * K16 * K23 + K4 * K9 * K15 * K21 + K4 * K9 * K16 * K18 - K4 * K9 * K16 * K22 + K4 * K10 * K15 * K18 - K4 * K10 * K15 * K22 - K4 * K14 * K15 * K22 - K4 * K14 * K16 * K17 + 2 * K7 * K8 * K15 * K23 + K7 * K9 * K15 * K18 - K7 * K9 * K15 * K22 + K7 * K9 * K16 * K17 + K7 * K10 * K15 * K17 - K7 * K14 * K15 * K17 - K8 * K9 * K15 * K21 - K8 * K9 * K16 * K18 + K8 * K9 * K16 * K22 - K8 * K10 * K15 * K18 + K8 * K10 * K15 * K22 - K8 * K17 * K18 - K8 * K17 * K22 - K8 * K14 * K15 * K17 + K8 * K14 * K16 * K17 + 4 * K9 * K10 * K15 * K16 + 2 * K9 * K13 * K15 * K22 - 4 * K9 * K14 * K15 * K16 - 2 * K10 * K14 * K15 * K22 - K15 * K24 * K4 * K22 - K15 * K24 * K8 * K22 + K15 * K28 * K4 * K22 + K15 * K28 * K8 * K22 - 2 * K16 * K23 * K8 * K22 - K16 * K27 * K3 * K22 + 2 * K23 * K27 * K1 * K22 - 2 * K24 * K28 * K1 * K22 + K1 * K2 * K24 * K22 + K1 * K2 * K28 * K22 + K2 * K2 * K23 * K22 + K9 * K2 * K16 * K22 + K10 * K2 * K15 * K22 + K15 * K2 * K14 * K22 + K15 * K22$

TABLE VII (continued)

T32 = 8*K1*K2*K2*K23*K27-8*K1*K2*K24*K28-4*K1*K2*K23**2+6*K1*K2*K24
2+2*K1*K2*K202+K1*K3*K18*K28-K1*K3*K20*K27-K1*K3*K21*K23+K1*K3*K21*K
K27+K1*K3*K22*K24-2*K1*K3*K22*K28+K1*K4*K17*K28-K1*K4*K18*K23+3*K1*K4*K
K18*K27-K1*K4*K20*K24+K1*K4*K20*K28+2*K1*K4*K21*K24-K1*K4*K21*K28+2*K1*K
K4*K22*K23-2*K1*K4*K22*K27-K1*K6*K17*K27-K1*K6*K18*K24+K1*K6*K18*K28-K1
*K6*K18*K23+K1*K6*K22*K24-K1*K6*K22*K28-K1*K7*K17*K23+K1*K7*K17*K27+2*K
K1*K7*K18*K24-K1*K7*K18*K28-K1*K7*K20*K23-K1*K7*K22*K24+K1*K8*K17*K24-2
*K1*K8*K17*K28+2*K1*K8*K18*K23-2*K1*K8*K18*K27+K1*K8*K20*K24-K1*K8*K20*K
K28-K1*K8*K21*K24-3*K1*K8*K22*K23+K1*K8*K22*K27+4*K1*K9*K15*K24-6*K1*K9
*K15*K28+4*K1*K9*K16*K23-4*K1*K9*K16*K27+2*K1*K9*K17*K21-4*K1*K9*K18*K
K22+2*K1*K9*K20*K21+K1*K9*K18*K22+2*K1*K9*K22**2+4*K1*K10*K15*K23-4*K1*K
K10*K15*K27-6*K1*K10*K16*K24+4*K1*K10*K16*K28-2*K1*K10*K17*K18+2*K1*K10
*K18*K20-4*K1*K10*K18*K21-2*K1*K10*K20*K22+2*K1*K10*K21*K22-4*K1*K13*K
K15*K24+2*K1*K13*K15*K28-4*K1*K13*K16*K23+2*K1*K13*K17*K20-2*K1*K13*K17
*K21+4*K1*K13*K18*K22-3*K1*K13*K18**2-K1*K13*K22**2-6*K1*K14*K15*K23+2*K
K1*K14*K15*K27+4*K1*K14*K16*K24-2*K1*K14*K16*K28+2*K1*K14*K17*K22-2*K1*K
K14*K18*K20+2*K1*K14*K18*K21+2*K1*K14*K20*K22+2*K2*K3*K17*K24+3*K2*K3*K18
*K27-K2*K3*K20*K24+K2*K3*K20*K28+2*K2*K3*K21*K24-K2*K3*K21*K28+K2*K3*K
K22*K23-2*K2*K3*K22*K27+3*K2*K4*K17*K27+4*K2*K4*K18*K24-3*K2*K4*K18*K28
-K2*K4*K20*K23+2*K2*K4*K21*K23-3*K2*K4*K22*K24+2*K2*K4*K22*K28-K2*K4*K6*K
K17*K24+K2*K6*K17*K28-K2*K6*K18*K23+K2*K6*K22*K23+2*K2*K7*K17*K24-K2*K7*K
K17*K28+2*K2*K7*K18*K23-K2*K7*K22*K23+K2*K8*K17*K23-2*K2*K8*K17*K27-3*K
K2*K8*K18*K24+2*K2*K8*K18*K28+K2*K8*K20*K23-K2*K8*K21*K23+2*K2*K8*K22*K
K24-K2*K8*K22*K28+4*K2*K9*K15*K23-4*K2*K9*K15*K27-6*K2*K9*K16*K24+4*K2*K
K9*K16*K28-2*K2*K9*K17*K22+2*K2*K9*K18*K20-4*K2*K9*K18*K21-2*K2*K9*K20*K
K22+2*K2*K9*K21*K22-6*K2*K10*K15*K24+4*K2*K10*K15*K28-6*K2*K10*K16*K23+
2*K2*K10*K17*K20+4*K2*K10*K17*K21+6*K2*K10*K18*K22-K2*K10*K17**2-4*K2*K
K10*K18**2-2*K2*K10*K22**2-4*K2*K21*K15*K23-6*K2*K21*K13*K17*K18+4*K2*K21*K
K17*K22+4*K2*K21*K14*K15*K24-2*K2*K21*K14*K15*K28+4*K2*K21*K16*K23-2*K2*K21*K
K20+2*K2*K21*K14*K17*K21-4*K2*K21*K14*K18*K22+3*K2*K21*K14*K22**2+K2*K21*K
K3*K4*K15*K28-6*K3*K4*K6*K16*K27+2*K3*K6*K15*K27+2*K3*K6*K16*K24-2*K3*K6*K
K16*K28+2*K3*K7*K15*K23-2*K3*K7*K15*K27-4*K3*K7*K16*K24+2*K3*K7*K16*K28-
2*K3*K8*K15*K24+K3*K4*K3*K8*K15*K28-2*K3*K8*K16*K23+4*K3*K8*K16*K27-K3*K9*K
K15*K21+K3*K9*K16*K22+K3*K10*K15*K22+K3*K10*K16*K17-K3*K10*K16*K20+2*K3
*K10*K16*K21-K3*K13*K15*K20+K3*K13*K15*K21+3*K3*K13*K16*K18-2*K3*K13*K
K16*K22+K3*K14*K15*K18-2*K3*K14*K15*K22+K3*K14*K16*K20-K3*K14*K16*K21+2
*K4*K6*K15*K24-2*K4*K6*K15*K28+2*K4*K6*K16*K23-4*K4*K7*K15*K24+2*K4*K7*K
K15*K28-4*K4*K7*K16*K23-4*K4*K8*K15*K23+4*K4*K8*K15*K27+6*K4*K8*K16*K24-
4*K4*K8*K16*K28-K4*K9*K15*K18+2*K4*K9*K15*K22-K4*K9*K16*K20+2*K4*K9*K
K16*K21-K4*K9*K15*K20+2*K4*K10*K15*K21+4*K4*K10*K16*K17+3*K4*K10*K16*K17
K22+3*K4*K13*K15*K18-2*K4*K13*K15*K22+3*K4*K13*K16*K17+4*K4*K14*K15*K17+
K4*K14*K15*K20-K4*K14*K15*K21-3*K4*K14*K16*K18+2*K4*K14*K16*K22+2*K6*K7
*K15*K23-2*K6*K8*K15*K24+2*K6*K8*K15*K28-2*K6*K8*K16*K23-K5*K9*K15*K21-K
K6*K9*K16*K18+K6*K9*K16*K21-K6*K10*K15*K18+K6*K10*K15*K22-K6*K10*K16*K
K17-K6*K13*K15*K17+K6*K14*K15*K18-K6*K14*K15*K22+K6*K14*K16*K17+2*K7*K8
*K15*K24+2*K7*K8*K16*K23-K7*K9*K15*K17-K7*K9*K15*K20+2*K7*K9*K16*K18-K7
*K9*K16*K22+2*K7*K10*K15*K18-K7*K10*K15*K22+2*K7*K10*K16*K17+K7*K13*K15
*K17-K7*K14*K15*K18-K7*K14*K16*K17+2*K8*K9*K15*K18-3*K8*K9*K15*K22+K8*K
K9*K16*K17+K8*K9*K16*K20-K8*K9*K16*K21+K8*K10*K15*K17+K8*K10*K15*K20-K8
*K10*K15*K21-3*K8*K10*K16*K18+2*K8*K10*K16*K22-2*K8*K13*K15*K18+K8*K13*K
K15*K22-2*K8*K13*K16*K17-2*K8*K14*K15*K17-K8*K14*K15*K20+2*K8*K14*K16*K
K18-K8*K14*K16*K22-4*K9*K10*K15**2+6*K9*K10*K16**2+8*K9*K13*K15*K16+6*K
K9*K14*K15**2-4*K9*K16*K16**2+6*K15*K16*K10**2+2*K15*K16*K14**2+K15*K23*K4
2+3*K15*K23*K82-3*K15*K27*K4**2-K15*K27*K8**2-K16*K24*K3**2-4*K16*K
K24*K4**2-2*K16*K24*K8**2+3*K16*K28*K4**2+K16*K28*K8**2-4*K23*K24*K1**2
+6*K23*K24*K2**2+6*K23*K28*K1**2-4*K23*K28*K2**2+4*K24*K27*K1**2-2*K27*K
K28*K1**2

TABLE VII (continued)

$$\begin{aligned}
 T31 = & -8 * K1 * K2 * K23 * K24 + 8 * K1 * K2 * K23 * K28 + 12 * K1 * K2 * K24 * K27 - 4 * K1 * K2 * \\
 & K27 * K28 - K1 * K3 * K20 * K28 - K1 * K3 * K22 * K23 + K1 * K3 * K22 * K27 + 2 * K1 * K4 * K18 * K28 + K1 * K4 * \\
 & * K20 * K23 - 2 * K1 * K4 * K20 * K27 - K1 * K4 * K21 * K23 + K1 * K4 * K21 * K27 + 2 * K1 * K4 * K22 * K24 - 2 * \\
 & K1 * K4 * K22 * K28 - K1 * K6 * K17 * K28 + K1 * K6 * K18 * K23 - 2 * K1 * K6 * K18 * K27 + K1 * K6 * K20 * K24 \\
 & - K1 * K6 * K20 * K28 - K1 * K6 * K21 * K24 - 2 * K1 * K6 * K22 * K23 + K1 * K6 * K22 * K27 - K1 * K7 * K18 * \\
 & K23 + K1 * K7 * K18 * K27 - K1 * K7 * K20 * K24 - K1 * K8 * K17 * K23 + K1 * K8 * K17 * K27 + 2 * K1 * K8 * K18 * \\
 & * K24 - 2 * K1 * K8 * K18 * K28 - 2 * K1 * K8 * K20 * K23 + K1 * K8 * K20 * K27 - 2 * K1 * K8 * K22 * K24 - 6 * K1 * \\
 & * K9 * K15 * K23 + 4 * K1 * K9 * K15 * K27 + 4 * K1 * K9 * K16 * K24 - 4 * K1 * K9 * K16 * K28 + 2 * K1 * K9 * K17 * \\
 & * K22 - 2 * K1 * K9 * K18 * K20 + 2 * K1 * K9 * K18 * K21 + 4 * K1 * K9 * K20 * K22 + 4 * K1 * K10 * K15 * K24 - 4 * \\
 & * K1 * K10 * K15 * K28 + 4 * K1 * K10 * K16 * K23 - 6 * K1 * K10 * K16 * K27 + 2 * K1 * K10 * K17 * K20 - 2 * K1 * \\
 & * K10 * K18 * K22 + 2 * K1 * K10 * K20 * K21 - 2 * K1 * K10 * K18 * K22 - K1 * K10 * K20 * K22 + 2 * K1 * K10 * \\
 & K22 * K24 + 4 * K1 * K13 * K15 * K23 - 2 * K1 * K13 * K15 * K27 - 6 * K1 * K13 * K16 * K24 + 2 * K1 * K13 * K16 * \\
 & K28 - 2 * K1 * K13 * K17 * K22 + 4 * K1 * K13 * K18 * K20 - 2 * K1 * K13 * K18 * K21 - 2 * K1 * K13 * K20 * K22 - \\
 & - 4 * K1 * K14 * K15 * K24 - 4 * K1 * K14 * K16 * K23 + 2 * K1 * K14 * K16 * K27 + 2 * K1 * K14 * K18 * K22 + K1 * \\
 & * K14 * K20 * K22 + 2 * K2 * K3 * K17 * K27 + 3 * K2 * K3 * K18 * K24 - 2 * K2 * K3 * K20 * K27 + K2 * K3 * K21 * \\
 & K27 - K2 * K3 * K22 * K28 + 3 * K2 * K4 * K17 * K24 + 6 * K2 * K4 * K18 * K27 - 3 * K2 * K4 * K20 * K24 + 2 * K2 * \\
 & K4 * K20 * K28 + 3 * K2 * K4 * K21 * K24 - K2 * K4 * K21 * K28 + 2 * K2 * K4 * K22 * K23 - 3 * K2 * K4 * K22 * \\
 & K27 - 2 * K2 * K6 * K17 * K27 - 3 * K2 * K6 * K16 * K24 + 2 * K2 * K6 * K18 * K28 + K2 * K6 * K20 * K23 - K2 * K6 * \\
 & * K21 * K23 + 2 * K2 * K6 * K22 * K24 - K2 * K6 * K22 * K28 + K2 * K7 * K17 * K27 + 3 * K2 * K7 * K18 * K24 - K2 * \\
 & * K7 * K18 * K28 - K2 * K7 * K20 * K23 - K2 * K7 * K22 * K24 - K2 * K8 * K17 * K28 + 2 * K2 * K8 * K18 * K23 - 3 * \\
 & * K2 * K8 * K18 * K27 + 2 * K2 * K8 * K20 * K24 - K2 * K8 * K20 * K28 - K2 * K8 * K21 * K24 - 2 * K2 * K8 * K22 * \\
 & K23 + K2 * K8 * K22 * K27 + 4 * K2 * K9 * K15 * K24 - 4 * K2 * K9 * K15 * K28 + 4 * K2 * K9 * K16 * K23 - 6 * K2 * \\
 & * K9 * K16 * K27 - 4 * K2 * K9 * K18 * K22 + 2 * K2 * K9 * K20 * K21 - K2 * K9 * K20 * K22 + 2 * K2 * K9 * K22 * K22 + \\
 & 4 * K2 * K10 * K15 * K23 - 6 * K2 * K10 * K15 * K27 - 12 * K2 * K10 * K16 * K24 + 6 * K2 * K10 * K16 * K28 - 6 * \\
 & K2 * K10 * K17 * K18 + 6 * K2 * K10 * K18 * K20 - 6 * K2 * K10 * K18 * K21 - 4 * K2 * K10 * K20 * K22 + 2 * K2 * \\
 & K10 * K21 * K22 - 6 * K2 * K13 * K15 * K24 + 2 * K2 * K13 * K15 * K28 - 6 * K2 * K13 * K16 * K23 + 4 * K2 * K13 * \\
 & * K17 * K20 - 2 * K2 * K13 * K17 * K21 + 6 * K2 * K13 * K18 * K22 - 2 * K2 * K13 * K17 * K22 - 6 * K2 * K13 * K18 * \\
 & * K22 + 2 * K2 * K13 * K22 * K24 - 4 * K2 * K14 * K15 * K23 + 2 * K2 * K14 * K15 * K27 + 6 * K2 * K14 * K16 * K24 - 2 * \\
 & K2 * K14 * K16 * K28 + 2 * K2 * K14 * K17 * K22 - 4 * K2 * K14 * K18 * K20 + 2 * K2 * K14 * K18 * K21 + 2 * K2 * \\
 & K14 * K20 * K22 - 6 * K3 * K4 * K16 * K24 + 2 * K3 * K6 * K15 * K28 + 4 * K3 * K6 * K16 * K27 - 2 * K3 * K7 * K16 * \\
 & K27 + 2 * K3 * K8 * K15 * K23 - 2 * K3 * K8 * K15 * K27 + 2 * K3 * K8 * K16 * K28 - K3 * K9 * K15 * K22 + 3 * K3 * \\
 & * K10 * K16 * K16 * K3 * K13 * K15 * K22 + 2 * K3 * K13 * K16 * K17 - 2 * K3 * K13 * K16 * K20 + K3 * K13 * \\
 & K16 * K21 - K3 * K14 * K15 * K20 - K3 * K14 * K16 * K22 - 2 * K4 * K6 * K15 * K23 + 4 * K4 * K6 * K15 * K27 + 6 * \\
 & * K4 * K6 * K16 * K24 - 4 * K4 * K6 * K16 * K28 + 2 * K4 * K7 * K15 * K23 - 2 * K4 * K7 * K15 * K27 - 6 * K4 * K7 * \\
 & K16 * K24 + 2 * K4 * K7 * K16 * K28 - 4 * K4 * K8 * K15 * K24 + 4 * K4 * K8 * K15 * K28 - 4 * K4 * K8 * K16 * K23 \\
 & + 6 * K4 * K8 * K16 * K27 + K4 * K9 * K15 * K20 - K4 * K9 * K15 * K21 + 2 * K4 * K9 * K16 * K22 + 2 * K4 * K9 * K10 * \\
 & K15 * K22 + 3 * K4 * K10 * K16 * K17 - 3 * K4 * K10 * K16 * K20 + 3 * K4 * K10 * K16 * K21 - 2 * K4 * K13 * K15 * \\
 & K20 + K4 * K13 * K15 * K21 + 6 * K4 * K13 * K16 * K18 - 3 * K4 * K13 * K16 * K22 + 2 * K4 * K14 * K15 * K18 - \\
 & 2 * K4 * K14 * K15 * K22 + 2 * K4 * K14 * K16 * K20 - K4 * K14 * K16 * K21 + 2 * K6 * K7 * K15 * K24 + 2 * K6 * \\
 & K7 * K16 * K23 + 4 * K6 * K8 * K15 * K23 - 2 * K6 * K8 * K15 * K27 - 4 * K6 * K8 * K16 * K24 + 2 * K6 * K8 * K16 * \\
 & K28 + K6 * K9 * K15 * K18 - 2 * K6 * K9 * K15 * K22 + K6 * K9 * K16 * K20 - K6 * K9 * K16 * K21 + K6 * K10 * \\
 & K15 * K20 - K6 * K10 * K15 * K21 - 3 * K6 * K10 * K16 * K18 + 2 * K6 * K10 * K16 * K22 - 2 * K6 * K13 * K15 * \\
 & K18 + K6 * K13 * K15 * K22 - 2 * K6 * K13 * K16 * K17 - K6 * K14 * K15 * K17 - K6 * K14 * K15 * K20 + 2 * K6 * \\
 & K14 * K16 * K18 - K6 * K14 * K16 * K22 + 2 * K7 * K8 * K16 * K24 - K7 * K9 * K15 * K18 - K7 * K9 * K16 * K20 - \\
 & K7 * K10 * K15 * K20 + 3 * K7 * K10 * K16 * K18 - K7 * K10 * K16 * K22 + K7 * K13 * K15 * K18 + K7 * K13 * \\
 & K16 * K17 - K7 * K14 * K16 * K18 - K7 * K9 * K15 * K17 - 2 * K8 * K9 * K15 * K20 + 2 * K8 * K9 * K16 * K18 - 2 * \\
 & K8 * K9 * K16 * K22 + 2 * K8 * K10 * K15 * K18 - 2 * K8 * K10 * K15 * K22 + 2 * K8 * K10 * K16 * K20 - K8 * K10 * \\
 & * K16 * K21 + K8 * K13 * K15 * K17 + K8 * K13 * K15 * K20 - 3 * K8 * K13 * K16 * K18 + K8 * K13 * K16 * K22 - \\
 & 2 * K8 * K14 * K15 * K18 - K8 * K14 * K16 * K17 - K8 * K14 * K16 * K20 - 8 * K9 * K10 * K15 * K16 - 4 * K9 * \\
 & K13 * K15 * K22 + 6 * K9 * K13 * K16 * K22 + 8 * K9 * K14 * K15 * K16 + 12 * K10 * K13 * K15 * K16 + 4 * K10 * \\
 & K14 * K15 * K22 - 6 * K10 * K14 * K16 * K22 - 4 * K13 * K14 * K15 * K16 - K15 * K24 * K6 * K22 + 2 * K15 * K24 * \\
 & K8 * K22 - 2 * K15 * K26 * K4 * K22 + K15 * K28 * K6 * K22 - K16 * K23 * K6 * K22 + 2 * K15 * K23 * K8 * K22 - 2 * K16 * \\
 & * K27 * K3 * K22 - 6 * K16 * K27 * K4 * K22 - K16 * K27 * K8 * K22 - 4 * K23 * K27 * K1 * K22 + 6 * K23 * K27 * K2 * K22 \\
 & + 4 * K24 * K28 * K1 * K22 - 6 * K24 * K28 * K2 * K22 + 3 * K1 * K22 * K2 * K22 - 2 * K1 * K22 * K24 * K22 + K1 * K22 * \\
 & K27 * K22 - 2 * K2 * K23 * K22 + 6 * K2 * K24 * K22 + K2 * K28 * K22 + 3 * K9 * K22 * K15 * K22 - 2 * K9 * \\
 & 2 * K16 * K22 * K10 * K22 * K15 * K22 + 6 * K10 * K22 * K16 * K22 + K13 * K22 * K15 * K22 + 2 * K14 * K22 * K16 * K22
 \end{aligned}$$

TABLE VII (continued)

$$T30 = -4 * K1 * K2 * K23 * K27 + 4 * K1 * K2 * K24 * K28 + 2 * K1 * K2 * K23 * K27 + 2 * K1 * K2 * K24 * K28 - K1 * K4 * K22 * K23 + K1 * K4 * K22 * K27 - K1 * K6 * K18 * K28 - K1 * K6 * K20 * K23 + K1 * K6 * K20 * K27 - K1 * K6 * K22 * K24 - K1 * K8 * K18 * K23 + K1 * K8 * K18 * K27 - K1 * K8 * K20 * K24 - 2 * K1 * K9 * K15 * K24 - 2 * K1 * K9 * K16 * K23 + 2 * K1 * K9 * K16 * K27 + 2 * K1 * K9 * K18 * K22 + K1 * K9 * K20 * K22 - 2 * K1 * K10 * K15 * K23 + 2 * K1 * K10 * K15 * K27 + 2 * K1 * K10 * K16 * K24 - 2 * K1 * K10 * K16 * K28 + 2 * K1 * K10 * K17 * K18 + 2 * K1 * K10 * K20 * K22 + 2 * K1 * K13 * K15 * K24 + 2 * K1 * K13 * K16 * K23 - 2 * K1 * K13 * K16 * K27 - 2 * K1 * K13 * K18 * K22 - K1 * K13 * K20 * K22 - 4 * K1 * K14 * K16 * K24 - 2 * K1 * K14 * K17 * K18 + 2 * K1 * K14 * K18 * K22 + 2 * K2 * K3 * K17 * K24 + 3 * K2 * K3 * K18 * K27 - 2 * K2 * K3 * K20 * K24 + K2 * K3 * K21 * K24 + 3 * K2 * K4 * K17 * K27 + 6 * K2 * K4 * K18 * K24 - 3 * K2 * K4 * K20 * K27 + K2 * K4 * K21 * K27 - K2 * K4 * K22 * K28 - 2 * K2 * K6 * K17 * K24 - 3 * K2 * K6 * K18 * K27 + 2 * K2 * K6 * K20 * K24 - K2 * K6 * K20 * K28 - K2 * K6 * K21 * K24 - K2 * K6 * K22 * K23 + K2 * K6 * K22 * K27 + K2 * K7 * K17 * K24 + K2 * K7 * K18 * K27 - K2 * K7 * K20 * K24 - K2 * K8 * K18 * K28 - K2 * K8 * K20 * K23 + K2 * K8 * K20 * K27 - K2 * K8 * K22 * K24 - 2 * K2 * K9 * K15 * K23 + 2 * K2 * K9 * K15 * K27 - 2 * K2 * K9 * K16 * K28 + 2 * K2 * K9 * K20 * K22 - 2 * K2 * K10 * K15 * K28 - 3 * K2 * K10 * K16 * K27 + 4 * K2 * K10 * K17 * K20 - 2 * K2 * K10 * K17 * K21 + 2 * K2 * K10 * K20 * K21 - 2 * K2 * K10 * K17 * K22 - 6 * K2 * K10 * K18 * K22 - 2 * K2 * K10 * K20 * K22 + K2 * K10 * K22 * K22 + 2 * K2 * K13 * K15 * K23 - 2 * K2 * K13 * K15 * K27 - 8 * K2 * K13 * K16 * K24 + 2 * K2 * K13 * K16 * K28 - 6 * K2 * K13 * K17 * K18 + 6 * K2 * K13 * K18 * K20 - 2 * K2 * K13 * K18 * K21 - 2 * K2 * K13 * K20 * K22 - 2 * K2 * K14 * K15 * K24 - 2 * K2 * K14 * K16 * K23 + 2 * K2 * K14 * K16 * K27 + 2 * K2 * K14 * K18 * K22 + K2 * K14 * K20 * K22 - 6 * K3 * K4 * K16 * K27 + 4 * K3 * K6 * K16 * K24 - 2 * K3 * K7 * K16 * K24 + 2 * K3 * K10 * K17 - 2 * K3 * K10 * K16 * K20 + K3 * K10 * K16 * K21 + 3 * K3 * K13 * K16 * K18 + 2 * K4 * K6 * K15 * K23 + 6 * K4 * K6 * K16 * K27 - 2 * K4 * K7 * K16 * K27 + 2 * K4 * K8 * K15 * K23 - 2 * K4 * K8 * K15 * K27 + 2 * K4 * K8 * K16 * K28 - K4 * K9 * K15 * K22 + 6 * K4 * K10 * K15 * K18 + K4 * K13 * K15 * K22 + 3 * K4 * K13 * K16 * K17 - 3 * K4 * K13 * K16 * K20 + K4 * K13 * K16 * K21 - K4 * K14 * K15 * K20 - K4 * K14 * K16 * K22 + 2 * K6 * K7 * K16 * K24 + 2 * K6 * K8 * K15 * K24 + 2 * K6 * K8 * K16 * K23 - 2 * K6 * K8 * K16 * K27 - K6 * K9 * K15 * K20 - K6 * K9 * K16 * K22 - K6 * K10 * K15 * K22 - 2 * K6 * K10 * K16 * K17 + 2 * K6 * K10 * K16 * K20 - K6 * K10 * K16 * K21 + K6 * K13 * K15 * K20 - 3 * K6 * K13 * K16 * K18 + K6 * K13 * K16 * K22 - K6 * K14 * K15 * K18 - K6 * K14 * K16 * K20 + K7 * K10 * K16 * K17 - K7 * K10 * K20 * K7 * K13 * K16 * K18 - K8 * K9 * K15 * K18 - K8 * K9 * K16 * K20 - K8 * K10 * K15 * K20 - K8 * K10 * K16 * K22 + K8 * K13 * K15 * K18 + K8 * K13 * K16 * K20 - K8 * K14 * K16 * K18 + 2 * K9 * K10 * K15 * K22 - 4 * K9 * K13 * K15 * K16 + 2 * K9 * K14 * K16 * K22 - 2 * K10 * K13 * K15 * K22 + 8 * K10 * K13 * K16 * K22 + 4 * K10 * K14 * K15 * K15 * K16 - 2 * K13 * K14 * K16 * K22 + 2 * K15 * K16 * K9 * K22 + 2 * K15 * K16 * K13 * K22 + K15 * K23 * K6 * K22 - K15 * K27 * K6 * K22 - 2 * K16 * K24 * K3 * K22 - 6 * K16 * K24 * K4 * K22 - 2 * K16 * K24 * K6 * K22 + K16 * K24 * K8 * K22 + K16 * K28 * K6 * K22 + 2 * K23 * K24 * K1 * K22 + 2 * K23 * K28 * K2 * K22 - 2 * K24 * K27 * K1 * K22 + 8 * K24 * K27 * K2 * K22 - 2 * K27 * K28 * K2 * K22$$

$$T43 = -4 * K1 * K2 * K23 * K27 + 4 * K1 * K2 * K24 * K28 - 2 * K1 * K2 * K23 * K27 - 2 * K1 * K2 * K24 * K28 - K1 * K3 * K21 * K27 - K1 * K4 * K21 * K27 - K1 * K4 * K21 * K24 + K1 * K4 * K21 * K28 + K1 * K4 * K22 * K27 - K1 * K7 * K17 * K27 - K1 * K7 * K18 * K24 + K1 * K7 * K18 * K28 - K1 * K7 * K21 * K23 + K1 * K7 * K22 * K24 - K1 * K7 * K22 * K28 + K1 * K8 * K18 * K27 + K1 * K8 * K21 * K24 - K1 * K8 * K21 * K28 - K1 * K8 * K22 * K27 + 2 * K1 * K9 * K16 * K27 + K1 * K9 * K21 * K22 + 2 * K1 * K10 * K15 * K27 + 2 * K1 * K10 * K16 * K24 - 2 * K1 * K10 * K16 * K28 + 2 * K1 * K10 * K18 * K21 - 2 * K1 * K10 * K21 * K22 + 2 * K1 * K13 * K15 * K24 - 2 * K1 * K13 * K15 * K28 + 2 * K1 * K13 * K16 * K23 + 2 * K1 * K13 * K17 * K21 - 2 * K1 * K13 * K18 * K22 + K1 * K13 * K18 * K22 + K1 * K13 * K22 * K22 - 2 * K1 * K14 * K15 * K27 - 2 * K1 * K14 * K16 * K24 + 2 * K1 * K14 * K16 * K28 - 2 * K1 * K14 * K18 * K21 + 2 * K1 * K14 * K21 * K22 - K2 * K3 * K18 * K27 - K2 * K3 * K21 * K24 + K2 * K3 * K21 * K28 + K2 * K3 * K22 * K27 - K2 * K4 * K17 * K27 - K2 * K4 * K18 * K24 + K2 * K4 * K18 * K28 - K2 * K4 * K21 * K23 + K2 * K4 * K22 * K24 - K2 * K4 * K22 * K28 - K2 * K7 * K17 * K24 + K2 * K7 * K17 * K28 - K2 * K7 * K18 * K23 + K2 * K7 * K22 * K23 + K2 * K8 * K17 * K27 + K2 * K8 * K18 * K24 - K2 * K8 * K18 * K28 + K2 * K8 * K21 * K23 - K2 * K8 * K22 * K24 + K2 * K8 * K22 * K28 + 2 * K2 * K9 * K15 * K27 + 2 * K2 * K9 * K16 * K24 - 2 * K2 * K9 * K16 * K28 + 2 * K2 * K9 * K18 * K21 - 2 * K2 * K9 * K21 * K22 + 2 * K2 * K10 * K15 * K24 - 2 * K2 * K10 * K15 * K28 + 2 * K2 * K10 * K16 * K23 + 2 * K2 * K10 * K17 * K21 - 2 * K2 * K10 * K18 * K22 + K2 * K10 * K18 * K22 - 2 * K2 * K14 * K15 * K24 + 2 * K2 * K14 * K15 * K28 - 2 * K2 * K14 * K16 * K23 - 2 * K2 * K14 * K17 * K21 + 2 * K2 * K14 * K18 * K22 - K2 * K14 * K18 * K22 - K2 * K14 * K22 * K22 + 2 * K3 * K4 * K16 * K27 + 2 * K3 * K7 * K15 * K27 + 2 * K3 * K7 * K16 * K24 - 2 * K3 * K7 * K16 * K28 - 2 * K3 * K8 * K16 * K27 - K3 * K10 * K16 * K21 - K3 * K13 * K15 * K21 - K3 * K13 * K16 * K18 + K3 * K13 * K16 * K22 + K3 * K14 * K16 * K21 + 2 * K4 * K7 * K15 * K24 - 2 * K4 * K7 * K15 * K28 + 2 * K4 * K7 * K16 * K23 - 2 * K4 * K8 * K15 * K27 - 2 * K4 * K8 * K16 * K24 + 2 * K4 * K8 * K16 * K28 - K4 * K9 * K16 * K21 - K4 * K10 * K15 * K21 - K4 * K10 * K16 * K18 + K4 * K10 * K16 * K22 - K4 * K13 * K15 * K18 + K4 * K13 * K15 * K22 - K4 * K13 * K16 * K17 + K4 * K14 * K15 * K21 + K4 * K14 * K15 * K28 - 2 * K7 * K8 * K15 * K24 + 2 * K7 * K8 * K15 * K28 - 2 * K7 * K8 * K16 * K23 - K7 * K9 * K15 * K21 - K7 * K9 * K16 * K18 + K7 * K9 * K16 * K22 - K7 * K10 * K15 * K18 + K7 * K10 * K15 * K22 - K7 * K10 * K16 * K17 - K7 * K13 * K15 * K17 + K7 * K14 * K15 * K18 - K7 * K14 * K15 * K22 + K7 * K14 * K15 * K28 - K8 * K9 * K16 * K21 + K8 * K10 * K15 * K21 + K8 * K10 * K16 * K18 - K8 * K10 * K16 * K22 + K8 * K13 * K15 * K18 - K8 * K13 * K15 * K22 + K8 * K13 * K16 * K17 - K8 * K14 * K15 * K21 - K8 * K14 * K16 * K18 + K8 * K14 * K16 * K22 - 2 * K9 * K10 * K16 * K22 - 4 * K9 * K13 * K15 * K16 + 2 * K9 * K14 * K16 * K22 - 2 * K10 * K13 * K15 * K22 + 2 * K10 * K14 * K15 * K18 + 2 * K10 * K14 * K15 * K22 - 2 * K15 * K16 * K10 * K22 - 2 * K15 * K16 * K13 * K22 - 2 * K15 * K16 * K15 * K23 * K7 * K22 + K15 * K27 * K4 * K22 + K15 * K27 * K8 * K22 + K16 * K24 * K4 * K22 + K16 * K24 * K8 * K22 - K16 * K28 * K4 * K22 - K16 * K28 * K8 * K22 - 2 * K23 * K24 * K2 * K22 + 2 * K23 * K28 * K2 * K22 - 2 * K24 * K27 * K1 * K22 + 2 * K27 * K28 * K1 * K22$$

TABLE VII (continued)

$T_{42} = 4 * K1 * K2 * K23 * K24 - 8 * K1 * K2 * K23 * K28 - 12 * K1 * K2 * K24 * K27 + 8 * K1 * K2 * K27 * K28 - K1 * K3 * K22 * K27 - K1 * K4 * K18 * K28 + K1 * K4 * K22 * K27 + K1 * K4 * K21 * K23 - 2 * K1 * K4 * K21 * K27 - K1 * K4 * K22 * K24 + 2 * K1 * K4 * K22 * K28 + K1 * K6 * K18 * K27 + K1 * K6 * K21 * K24 - K1 * K6 * K21 * K28 - K1 * K6 * K22 * K27 - K1 * K7 * K17 * K28 + K1 * K7 * K18 * K23 - 2 * K1 * K7 * K18 * K27 + K1 * K7 * K20 * K24 - K1 * K7 * K20 * K28 - K1 * K7 * K21 * K24 - 2 * K1 * K7 * K22 * K23 + K1 * K7 * K22 * K27 - K1 * K8 * K17 * K27 - K1 * K8 * K18 * K24 + 2 * K1 * K8 * K18 * K28 - K1 * K8 * K20 * K27 - 2 * K1 * K8 * K21 * K23 + K1 * K8 * K21 * K27 + 2 * K1 * K8 * K22 * K24 - 3 * K1 * K8 * K22 * K28 - 4 * K1 * K9 * K15 * K27 - 2 * K1 * K9 * K16 * K24 + 4 * K1 * K9 * K16 * K28 - 2 * K1 * K9 * K18 * K21 + 4 * K1 * K9 * K21 * K22 - 2 * K1 * K10 * K15 * K24 + 4 * K1 * K10 * K15 * K28 - 2 * K1 * K10 * K16 * K23 + 6 * K1 * K10 * K16 * K27 + 2 * K1 * K10 * K18 * K22 - 2 * K1 * K10 * K20 * K21 + K1 * K10 * K21 * K22 - 2 * K1 * K10 * K22 * K23 - 4 * K1 * K13 * K15 * K23 + 4 * K1 * K13 * K15 * K27 + 6 * K1 * K13 * K16 * K24 - 4 * K1 * K13 * K16 * K28 + 2 * K1 * K13 * K17 * K28 - 2 * K1 * K13 * K18 * K20 + 4 * K1 * K13 * K18 * K21 + 2 * K1 * K13 * K20 * K22 - 2 * K1 * K13 * K21 * K22 + 4 * K1 * K14 * K15 * K24 - 6 * K1 * K14 * K15 * K28 + 4 * K1 * K14 * K16 * K23 - 4 * K1 * K14 * K16 * K27 + 2 * K1 * K14 * K17 * K21 - 4 * K1 * K14 * K18 * K22 + 2 * K1 * K14 * K20 * K21 + K1 * K14 * K18 * K22 + 3 * K1 * K14 * K22 * K23 - K2 * K3 * K17 * K27 - K2 * K3 * K18 * K24 + K2 * K3 * K20 * K27 - 2 * K2 * K3 * K21 * K27 + K2 * K3 * K22 * K28 - K2 * K4 * K17 * K24 - 4 * K2 * K4 * K18 * K27 + K2 * K4 * K20 * K24 - K2 * K4 * K22 * K28 - 3 * K2 * K4 * K21 * K24 + 2 * K2 * K4 * K21 * K28 - K2 * K4 * K22 * K23 + 3 * K2 * K4 * K22 * K27 + 2 * K2 * K6 * K17 * K27 + K2 * K6 * K18 * K24 - K2 * K6 * K18 * K28 + K2 * K6 * K21 * K23 - K2 * K6 * K22 * K24 + K2 * K6 * K22 * K28 - 2 * K2 * K7 * K17 * K27 - 3 * K2 * K7 * K18 * K24 + 2 * K2 * K7 * K18 * K28 + K2 * K7 * K20 * K23 - K2 * K7 * K21 * K23 + 2 * K2 * K7 * K22 * K24 - K2 * K7 * K22 * K28 + K2 * K8 * K17 * K28 - K2 * K8 * K18 * K23 + 3 * K2 * K8 * K18 * K27 - K2 * K8 * K20 * K24 + K2 * K8 * K20 * K28 + 2 * K2 * K8 * K21 * K24 - K2 * K8 * K21 * K28 + 2 * K2 * K8 * K22 * K23 - 2 * K2 * K8 * K22 * K27 - 2 * K2 * K9 * K15 * K24 + 4 * K2 * K9 * K15 * K28 - 2 * K2 * K9 * K16 * K23 + 6 * K2 * K9 * K16 * K27 + 2 * K2 * K9 * K18 * K22 - 2 * K2 * K9 * K20 * K21 + K2 * K9 * K21 * K22 - 2 * K2 * K9 * K22 * K23 + 6 * K2 * K10 * K15 * K23 + 6 * K2 * K10 * K15 * K27 + 8 * K2 * K10 * K16 * K24 - 6 * K2 * K10 * K16 * K28 + 2 * K2 * K10 * K17 * K18 - 2 * K2 * K10 * K18 * K20 + 6 * K2 * K10 * K18 * K21 + 2 * K2 * K10 * K20 * K22 - 4 * K2 * K10 * K21 * K22 + 6 * K2 * K13 * K15 * K24 - 4 * K2 * K13 * K15 * K28 + 6 * K2 * K13 * K16 * K23 - 2 * K2 * K13 * K17 * K20 + 4 * K2 * K13 * K17 * K21 - 6 * K2 * K13 * K18 * K22 + K2 * K13 * K17 * K22 + 4 * K2 * K13 * K18 * K22 + 2 * K2 * K13 * K22 * K23 + 4 * K2 * K14 * K15 * K23 - 4 * K2 * K14 * K15 * K27 - 6 * K2 * K14 * K16 * K24 + 4 * K2 * K14 * K16 * K28 - 2 * K2 * K14 * K17 * K22 + 2 * K2 * K14 * K18 * K20 - 4 * K2 * K14 * K18 * K21 - 2 * K2 * K14 * K20 * K22 + 2 * K2 * K14 * K21 * K22 + 2 * K3 * K4 * K16 * K24 - 2 * K3 * K6 * K16 * K27 + 2 * K3 * K7 * K15 * K28 + 4 * K3 * K7 * K16 * K27 + 2 * K3 * K8 * K15 * K27 - 2 * K3 * K8 * K16 * K28 - K3 * K10 * K16 * K18 - K3 * K13 * K15 * K22 - K3 * K13 * K16 * K17 + K3 * K13 * K16 * K20 - 2 * K3 * K13 * K15 * K21 - K3 * K14 * K15 * K21 + K3 * K14 * K16 * K22 - 2 * K4 * K6 * K15 * K27 - 2 * K4 * K6 * K16 * K24 + 2 * K4 * K6 * K16 * K28 - 2 * K4 * K7 * K15 * K23 + 4 * K4 * K7 * K15 * K27 + 6 * K4 * K7 * K16 * K24 - 4 * K4 * K7 * K16 * K28 + 2 * K4 * K8 * K15 * K24 - 4 * K4 * K8 * K15 * K28 + 2 * K4 * K8 * K16 * K23 - 6 * K4 * K8 * K16 * K27 + K4 * K9 * K15 * K21 - K4 * K9 * K16 * K22 - K4 * K10 * K15 * K22 - K4 * K10 * K16 * K17 + K4 * K10 * K16 * K20 - 3 * K4 * K10 * K16 * K21 + K4 * K13 * K15 * K20 - 2 * K4 * K13 * K15 * K21 - 4 * K4 * K13 * K16 * K18 + 3 * K4 * K13 * K16 * K22 - K4 * K14 * K15 * K18 + 2 * K4 * K14 * K15 * K22 - K4 * K14 * K16 * K20 + 2 * K4 * K14 * K16 * K21 - 2 * K6 * K7 * K15 * K24 + 2 * K6 * K7 * K15 * K28 - 2 * K6 * K7 * K16 * K23 + 2 * K6 * K8 * K15 * K21 + K6 * K10 * K16 * K18 - K6 * K10 * K16 * K22 + K6 * K13 * K15 * K18 - K6 * K13 * K15 * K22 + K6 * K13 * K16 * K17 - K6 * K14 * K15 * K21 - K6 * K14 * K16 * K18 + K6 * K14 * K16 * K22 + 4 * K7 * K8 * K15 * K23 - 2 * K7 * K8 * K15 * K27 - 4 * K7 * K8 * K16 * K24 + 2 * K7 * K8 * K16 * K28 + K7 * K9 * K15 * K18 - 2 * K7 * K9 * K15 * K22 + K7 * K9 * K16 * K20 - K7 * K9 * K16 * K21 + K7 * K10 * K15 * K21 - 3 * K7 * K10 * K16 * K18 + 2 * K7 * K10 * K16 * K22 - 2 * K7 * K13 * K15 * K18 + K7 * K13 * K15 * K22 - 2 * K7 * K13 * K16 * K17 - K7 * K14 * K15 * K17 - K7 * K14 * K15 * K20 + 2 * K7 * K14 * K16 * K18 - K7 * K14 * K16 * K22 - 2 * K8 * K9 * K15 * K21 - K8 * K9 * K16 * K18 + 2 * K8 * K9 * K16 * K22 - K8 * K10 * K15 * K18 + 2 * K8 * K10 * K15 * K22 - K8 * K10 * K16 * K20 + 2 * K8 * K10 * K16 * K21 - K8 * K13 * K15 * K20 + K8 * K13 * K15 * K21 + 3 * K8 * K13 * K16 * K18 - 2 * K8 * K13 * K16 * K22 + 2 * K8 * K14 * K15 * K18 - 3 * K8 * K14 * K15 * K22 + K8 * K14 * K16 * K17 + K8 * K14 * K16 * K20 - K8 * K14 * K16 * K21 + 4 * K9 * K10 * K15 * K16 + 4 * K9 * K10 * K15 * K22 + K9 * K13 * K15 * K16 - 12 * K10 * K13 * K15 * K16 - 4 * K10 * K14 * K15 * K22 + 6 * K10 * K14 * K16 * K22 + 3 * K13 * K14 * K15 * K16 + K15 * K24 * K7 * K22 - 2 * K15 * K24 * K8 * K22 + 2 * K15 * K28 * K4 * K22 + 3 * K15 * K28 * K3 * K22 + K16 * K23 * K7 * K22 - 2 * K16 * K23 * K8 * K22 + K16 * K27 * K3 * K22 + 4 * K16 * K27 * K4 * K22 + 2 * K16 * K27 * K8 * K22 + 4 * K16 * K23 * K27 * K1 * K22 - 6 * K23 * K27 * K1 * K22 - 4 * K24 * K28 * K1 * K22 + 6 * K24 * K28 * K2 * K22 + K1 * K22 * K24 * K22 - 2 * K1 * K22 * K27 * K22 + 3 * K1 * K22 * K28 * K22 + K2 * K22 * K23 * K22 - 4 * K22 * K24 * K22 - 2 * K22 * K28 * K22 + K9 * K22 * K16 * K22 + K10 * K22 * K15 * K22 - 4 * K10 * K22 * K16 * K22 - 2 * K13 * K22 * K15 * K22 + 3 * K14 * K22 * K15 * K22 - 2 * K14 * K22 * K16 * K22$

TABLE VII (continued)

T51 = -4*K1*K2*K23*K28+8*K1*K2*K27*K28+K1*K4*K22*K28-K1*K6*K21*K28-K1*K6*K22*K27-K1*K7*K20*K28-K1*K7*K22*K23+K1*K7*K22*K27+K1*K8*K18*K28-K1*K8*K20*K27-K1*K8*K21*K23+K1*K8*K21*K27+K1*K8*K22*K24-3*K1*K8*K22*K28-2*K1*K9*K15*K27+2*K1*K9*K16*K28+2*K1*K9*K21*K22+2*K1*K10*K15*K28-K1*K10*K22**2-2*K1*K13*K15*K23+4*K1*K13*K15*K27-4*K1*K13*K16*K28+2*K1*K13*K20*K22-2*K1*K13*K21*K22+2*K1*K14*K15*K24-6*K1*K14*K15*K28+2*K1*K14*K16*K23-4*K1*K14*K16*K27-2*K1*K14*K18*K22+2*K1*K14*K20*K21+3*K1*K14*K22**2+2*K1*K15*K17*K27+K2*K13*K18*K24-K2*K3*K20*K27+2*K2*K3*K21*K27+K2*K4*K17*K24+4*K2*K4*K18*K27-K2*K4*K20*K24+3*K2*K4*K21*K24-K2*K6*K17*K27-K2*K6*K18*K24+K2*K6*K20*K27-2*K2*K6*K21*K27+K2*K6*K22*K22+2*K2*K7*K17*K27+3*K2*K7*K18*K24-2*K2*K7*K20*K27+K2*K7*K21*K27-K2*K7*K22*K28+K2*K8*K20*K28-K2*K8*K21*K28+K2*K8*K22*K23-2*K2*K8*K22*K27+2*K2*K9*K15*K28-K2*K9*K22**2-3*K2*K10*K16*K24-2*K2*K10*K17*K18+2*K2*K10*K18*K20-6*K2*K10*K18*K21-4*K2*K13*K15*K28-2*K2*K13*K16*K27+2*K2*K13*K17*K20-4*K2*K13*K17*K21+4*K2*K13*K20*K21-K2*K13*K17**2-4*K2*K13*K18**2-K2*K13*K20**2-K2*K13*K21**2+2*K2*K13*K22**2+2*K2*K14*K15*K23-4*K2*K14*K15*K27+4*K2*K14*K16*K28-2*K2*K14*K20*K22+2*K2*K14*K21*K22-2*K3*K4*K16*K24+2*K3*K6*K16*K27-4*K3*K7*K16*K27+3*K10*K16*K18+3*K13*K16*K17-K3*K13*K16*K20+2*K3*K13*K16*K21+2*K4*K6*K16*K24-6*K4*K7*K16*K24-2*K4*K8*K15*K28+K4*K10*K16*K17-K4*K10*K16*K20+3*K4*K10*K16*K21+4*K4*K13*K16*K18+K4*K14*K15*K22+2*K6*K7*K15*K28+4*K6*K7*K16*K27+2*K6*K8*K15*K27-2*K6*K8*K16*K28-K6*K10*K16*K18-K6*K13*K15*K22-K6*K13*K16*K17+K6*K13*K16*K20-2*K6*K13*K16*K21-K6*K14*K15*K21+K6*K14*K16*K22+2*K7*K8*K15*K23-2*K7*K8*K15*K27+2*K7*K8*K16*K28-K7*K9*K15*K22+3*K7*K10*K16*K16+K7*K13*K15*K22+2*K7*K13*K16*K17-2*K7*K13*K16*K20+K7*K13*K16*K21-K7*K14*K15*K20-K7*K14*K16*K22-K8*K9*K15*K21+K8*K9*K16*K22+K8*K10*K16*K22-K8*K13*K15*K20+K8*K13*K15*K21-2*K8*K13*K16*K22+K8*K14*K15*K18-3*K8*K14*K15*K22+K8*K14*K16*K20-K8*K14*K16*K21+2*K9*K13*K15*K22-4*K9*K14*K15*K16-2*K10*K14*K15*K22+8*K13*K14*K15*K16-K15*K24*K8**2+3*K15*K28*K8**2-K16*K23*K8**2-K16*K27*K3**2-4*K16*K27*K4**2-K16*K27*K6**2-K16*K27*K7**2+2*K16*K27*K8**2+2*K23*K27*K1**2-2*K24*K28*K1**2-2*K1**2*K27**2+3*K1**2*K28**2+4*K2**2*K24**2+6*K2**2*K27**2-2*K2**2*K28**2+4*K10**2*K16**2-2*K13**2*K15**2+6*K13**2*K16**2+3*K14**2*K15**2-2*K14**2*K16**2

$$T50 = 2 * K1 * K2 * K20 ** 2 - K1 * K6 * K22 * K28 - K1 * K8 * K20 * K28 - K1 * K8 * K22 * K23 + K1 * K8 * K22 * K27 - 2 * K1 * K9 * K15 * K28 + K1 * K9 * K22 * ** 2 + 2 * K1 * K13 * K15 * K28 - K1 * K13 * K22 * ** 2 - 2 * K1 * K14 * K15 * K23 + 2 * K1 * K14 * K15 * K27 - 2 * K1 * K14 * K16 * K28 + 2 * K1 * K14 * K20 * K22 - K2 * K8 * K22 * K28 - 2 * K2 * K14 * K15 * K23 + 2 * K2 * K14 * K22 * ** 2 + 2 * K6 * K8 * K15 * K28 - K6 * K14 * K15 * K22 - K8 * K9 * K15 * K22 + K9 * K13 * K15 * K22 - K9 * K14 * K15 * K20 - K9 * K14 * K16 * K22 + 2 * K9 * K14 * K15 * ** 2 - 2 * K13 * K14 * K15 * ** 2 + 2 * K15 * K16 * K14 * ** 2 + K15 * K23 * K8 * ** 2 - K15 * K27 * K8 * ** 2 + K16 * K28 * K8 * ** 2 + 2 * K23 * K28 * K1 * ** 2 - 2 * K27 * K28 * K1 * ** 2$$
$$\begin{aligned} T63 = & -2*K1*K2*K27**2-K1*K7*K21*K27+2*K1*K13*K16*K27+K1*K13*K21 \\ & **2+K2*K7*K21*K28+K2*K7*K22*K27+K2*K8*K21*K27+2*K2*K13*K15*K27-2*K2*K13 \\ & *K16*K28-2*K2*K13*K21*K22-2*K2*K14*K16*K27-K2*K14*K21**2-2*K7*K8*K16* \\ & K27-K7*K13*K15*K21+K7*K13*K16*K22+K7*K14*K16*K21+K8*K13*K16*K21+2*K13* \\ & K14*K16**2-2*K15*K16*K13**2+K1**2*K7*K14-K1*K28*K7**2+2*K27*K28*K2**2 \end{aligned}$$

T62 = -4*K1*K2*K27*K28-K1*K7*K21*K28-K1*K7*K22*K27-1*K8*K21*K27-2*K1*K13*K15*K27+2*K1*K13*K16*K28+2*K1*K13*K21*K22+2*K1*K14*K16*K27+K1*K4*K21**2+K2*K7*K22*K2J+K2*K8*K21*K28+K2*K8*K22*K27+2*K2*K13*K15*K28-K2*K13*K22**2+2*K2*K14*K15*K27-2*K2*K14*K16*K28-2*K2*K14*K21*K22+2*K7*K8*K15*K27-2*K7*K8*K16*K28-K7*K13*K15*K22-K7*K14*K15*K21+K7*K14*K16*K22-K8*K13*K15*K21+K8*K13*K16*K22+K8*K14*K16*K21-4*K13*K14*K15*K16+K15*K28*K7*2-K16*K27*K8**2+K1**2*K27**2+K2**2*K28**2+K13**2*K15**2+K14**2*K16**2

$$T61 = -2 * K1 * K2 * K20 *** 2 - K1 * K7 * K22 * K28 - K1 * K8 * K21 * K28 - K1 * K8 * K22 * K27 - 2 * K1 * K13 * K15 * K28 + K1 * K13 * K22 *** 2 - 2 * K1 * K14 * K15 * K27 + 2 * K1 * K14 * K16 * K28 + 2 * K1 * K14 * K21 * K22 + K2 * K8 * K22 * K28 + 2 * K2 * K14 * K15 * K28 - K2 * K14 * K22 *** 2 + 2 * K7 * K8 * K15 * K28 - K7 * K14 * K15 * K22 - K8 * K13 * K15 * K22 - K8 * K14 * K15 * K21 + K8 * K14 * K16 * K22 + 2 * K13 * K14 * K15 *** 2 - 2 * K15 * K16 * K14 *** 2 + K15 * K27 * K8 *** 2 - K16 * K28 * K8 *** 2 + 2 * K27 * K28 * K1 *** 2$$
$$T_{60} = -K_1 * K_3 * K_{22} * K_{28} - 2 * K_1 * K_{14} * K_{15} * K_{28} + K_1 * K_{14} * K_{22} * K_{28} - K_8 * K_{14} * K_{15} * K_{22} + K_{15} * K_{28} * K_8 * K_{22} + K_1 * K_{28} * K_{15} * K_{22}$$

CHAPTER IV

KINEMATIC PAIRS

The previous chapter and Appendices B, C and D, completely define the direction cosines and locations of the Burmester lines in the moving and fixed frames of reference. These Burmester lines correspond to C - C cranks. Under certain conditions the C - C cranks may degenerate into other cranks containing R, P, H or C pairs.

The task of finding the conditions under which a C pair may act like a R, P, or H pair becomes simpler if we consider a new coordinate system x', y', z' in which the z' -axis coincides with the Burmester line in the fixed rigid body, and the x' -axis is the line which is perpendicular to this Burmester line and passes through a point P_1 on the Burmester line in the rigid body in its first position. The transformation of coordinates from the x, y, z frame of reference to the x', y', z' frame of reference is performed in the following section.

4.1. Coordinate Transformation

The coordinate transformation is performed in the following steps (see Figure 5):

- 1) Define the z' -axis.

We define the z' -axis to be collinear with the Burmester line in the fixed rigid body under consideration. This Burmester line can be expressed as

$$\frac{x - x_B}{l} = \frac{y - y_B}{m} = \frac{z - z_B}{n} \quad (4.1)$$

where (x_B, y_B, z_B) and (l, m, n) are the coordinates of a point on and the direction cosines of the Burmester line¹⁸.

2) Find plane $x'y'$.

Plane $x'y'$ must be perpendicular to the Burmester line in the fixed rigid body, and must also contain point P_1 with coordinates (x_1, y_1, z_1) . Point P_1 is the point on the Burmester line in the moving rigid body in its first position which is at the shortest distance from the fixed Burmester line. Points P_i , $i = 1, \dots, 5$ may be found by following the procedure outlined in Step 2, Section 3.2, Chapter III (there P' is our P_i). Plane $x'y'$ is of the form

$$A_{x'y'} x + B_{x'y'} y + C_{x'y'} z - 1 = 0 \quad (4.2)$$

The conditions for plane $x'y'$ to be perpendicular to the Burmester line in the fixed rigid body are:

$$A_{x'y'} m - B_{x'y'} l = 0 \quad (4.3)$$

and

$$A_{x'y'} n - C_{x'y'} l = 0 \quad (4.4)$$

The condition for a point P_1 to lie in the plane $x'y'$ is:

$$A_{x'y'} x_1 + B_{x'y'} y_1 + C_{x'y'} z_1 - 1 = 0 \quad (4.5)$$

Equations (4.3 - 4.5) form a system of three equations and three unknowns which can be solved simultaneously for $A_{x'y'}$, $B_{x'y'}$, and $C_{x'y'}$.

Thus, by defining

$$D_{x'y'} = \begin{vmatrix} m & -l & 0 \\ n & 0 & -l \\ x_1 & y_1 & z_1 \end{vmatrix}; \quad D_{A'} = \begin{vmatrix} 1 & -l & 0 \\ 0 & 0 & -l \\ 0 & y_1 & z_1 \end{vmatrix};$$

¹⁸ With respect to the x, y, z frame of reference.

$$D_{B'} = \begin{vmatrix} m & 1 & 0 \\ n & 0 & -\ell \\ x_1 & 0 & z_1 \end{vmatrix} ; \quad \text{and} \quad D_{C'} = \begin{vmatrix} m & -\ell & 1 \\ n & 0 & 0 \\ x_1 & y_1 & 0 \end{vmatrix} ,$$

we obtain

$$A_{x'y'} = \frac{D_{A'}}{D_{x'y'}} ; \quad B_{x'y'} = \frac{D_{B'}}{D_{x'y'}} ; \quad \text{and} \quad C_{x'y'} = \frac{D_{C'}}{D_{x'y'}} .$$

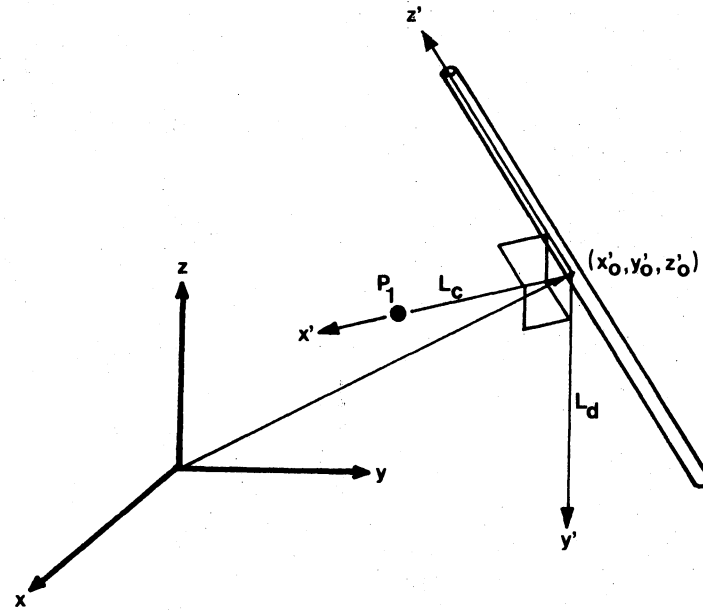


Figure 5. Transformation of coordinates $(x, y, z$ to $x', y', z')$

3) Find the point of intersection of the Burmester line with plane $x'y'$.

The point of intersection is:

$$(x_0', y_0', z_0') = (x_B + \ell r_0, y_B + m r_0, z_B + n r_0) \quad (4.6)$$

where

$$r_0 = \frac{-(A_{x'y'} x_B + B_{x'y'} y_B + C_{x'y'} z_B - 1)}{(A_{x'y'} \ell + B_{x'y'} m + C_{x'y'} n)} .$$

Notice that point (x_0', y_0', z_0') is the origin of the new frame of reference x', y', z' .

4) Define the x' -axis.

The x' -axis must be collinear with a line L_c passing through point

(x_0', y_0', z_0') and point $P_1 (x_1, y_1, z_1)$, thus

$$\frac{x - x_0'}{x_1 - x_0'} = \frac{y - y_0'}{y_1 - y_0'} = \frac{z - z_0'}{z_1 - z_0'}$$

or

$$\frac{x - x_0'}{l_c} = \frac{y - y_0'}{m_c} = \frac{z - z_0'}{n_c} \quad (4.7)$$

where (l_c, m_c, n_c) are the direction cosines of line L_c , and are defined as:

$$l_c = (x_1 - x_0') / s$$

$$m_c = (y_1 - y_0') / s$$

$$n_c = (z_1 - z_0') / s$$

$$s = \sqrt{(x_1 - x_0')^2 + (y_1 - y_0')^2 + (z_1 - z_0')^2} \quad , \quad (4.8)$$

here s is the distance between the origin of the frame of reference x', y', z' to point P_1 (s is also the shortest distance between the Burmester line in the fixed rigid body and the Burmester line in the moving rigid body in its first position).

5) Define the y' -axis.

The y' -axis must be collinear with a line L_d which is perpendicular to both the Burmester line in the fixed rigid body and line L_c at the same time. It must also pass through point (x_0', y_0', z_0') .

We can find the direction cosines (l_d, m_d, n_d) of line L_d from Appendix A, by using $\text{sgn} = -1$, and the Burmester line in the fixed rigid body with direction cosines (l, m, n) , and line L_c (l_c, m_c, n_c) as the intersecting lines. Line L_d is of the form

$$\frac{x - x_0'}{l_c} = \frac{y - y_0'}{m_c} = \frac{z - z_0'}{n_c} \quad . \quad (4.9)$$

4.2. Kinematic Inversion

In the previous section, we described a procedure to transform a

coordinate system from a x, y, z frame of reference to a new x', y', z' frame of reference, in which the z' -axis is coincident with the Burmester line in the fixed rigid body, and the x' -axis is perpendicular to this line and at the same time passes through a point P_1 , which is on the Burmester line in the moving rigid body in its first position.

Burmester theory deals strictly with relative displacements, i. e., the five finitely separated positions of a moving rigid body (Σ) relative to another fixed rigid body (Σ'); therefore, we can consider one of the positions of the moving rigid body (Σ_i , $i = 1, \dots$, or 5) as fixed, and then find the relative positions of Σ' with respect to the new fixed rigid body (Σ_i) by applying the principle of kinematic inversion.

In our case kinematic inversion is accomplished by performing a transformation of coordinates from a x, y, z frame of reference to a new x', y', z' frame of reference similar to the one performed in the previous section; but here the z' -axis is coincident with the Burmester line corresponding to Σ_i , $i = 1, \dots$, or 5, and the y' -axis is perpendicular to this Burmester line and at the same time it passes through a point P_1 which is on the Burmester line corresponding to Σ' in its first position relative to Σ_i .

4.3. Conditions Under Which a C Pair may be

Replaced by a R, P, or H Pair

In this section, we discuss the conditions under which a C pair may be replaced by a R, P or H pair. We know that a C pair has two degrees of freedom (a rotation about and a translation along the cylinder axis). If we constrain the translation along the cylinder axis, then the C pair behaves like a R pair. In the event that we constrain the rotation about

the cylinder axis, then the C pair behaves like a P pair. Finally, if we force the translation along the cylinder axis to be proportional to the rotation about the same cylinder axis, then the C pair behaves like a H pair.

In what follows, we give the conditions for a R, P, H and C kinematic pair to exist for five finitely separated positions of a rigid body.

1) Revolute Pair: all those points whose several positions can be reached with a R pair (i. e., points which fall on a circle) must satisfy the condition that for the five finitely separated positions of a rigid body, a point on Σ corresponding to Σ_j , $j = 1, \dots, 5$ must lie on a plane which is perpendicular to the Burmester line, see Figure 6.

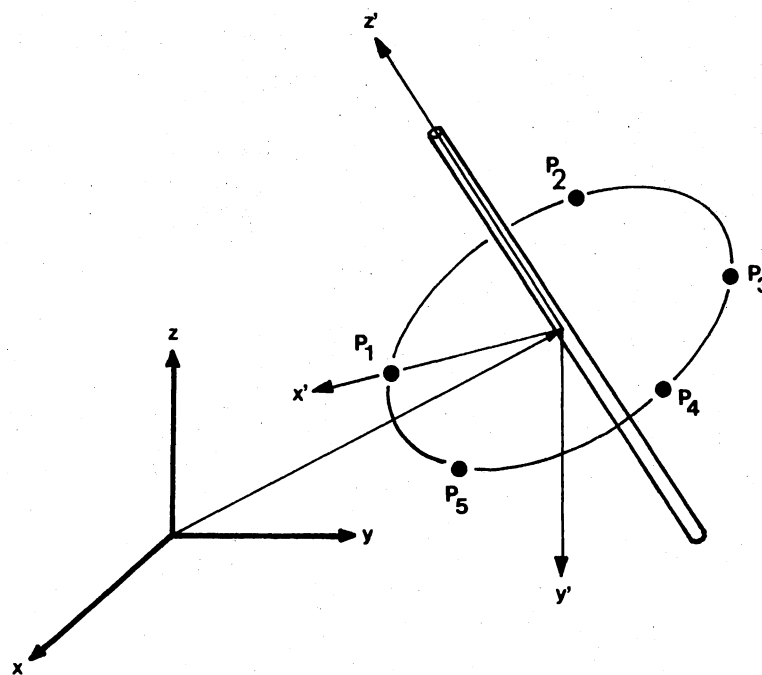


Figure 6. Points on a Circle

In order to satisfy this condition, a point on Σ in all of its five positions must lie on plane $x'y'$ of the x', y', z' frame of reference; i.e.,

the z' component of P_j (x_j', y_j', z_j'), $j = 1, \dots, 5$ must be zero. If this condition is not satisfied, then a R pair will not permit a rigid body to pass through the five prescribed positions.

2) Prismatic Pair: all those points whose several positions can be reached by a P pair must satisfy the condition that for the five finitely separated positions of a rigid body, a point on Σ corresponding to Σ_j , $j = 1, \dots, 5$ must lie on a straight line, see Figure 7.

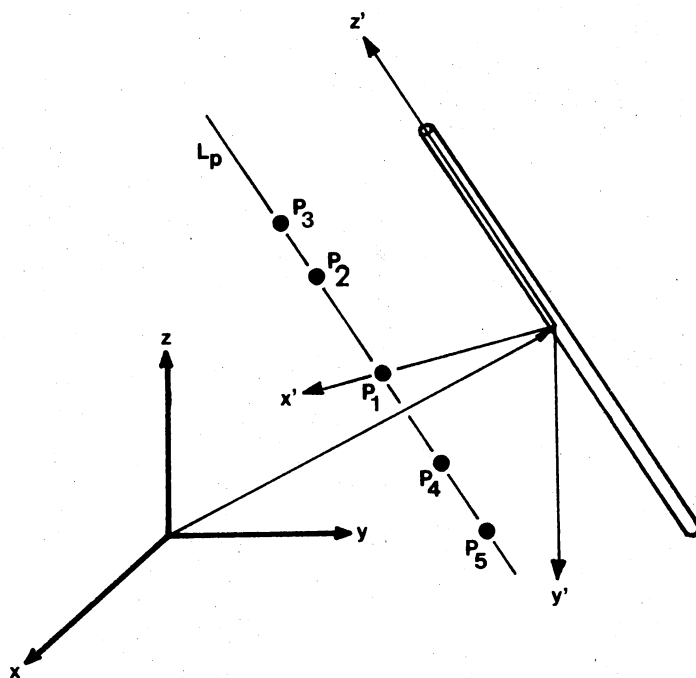


Figure 7. Points on a Straight Line

If this condition is not satisfied, then a P pair will not permit a rigid body to pass through the five prescribed positions.

3) Helical Pair: all those points whose several positions can be reached with a H pair must satisfy the condition that for the five finitely separated positions of a rigid body, a point on Σ corresponding to Σ_j , $j = 1, \dots, 5$ must lie on a right circular helix whose axis is the Burmester

line. To satisfy this condition, we must check if $\theta_j = \arccos x_j' / s = \arcsin y_j' / s$; if so, we calculate $C_{j-1} = \theta_j / z_j'$, $j = 2, \dots, 5$. If $C_1 = C_2 = C_3 = C_4$, then the condition is satisfied, see Figure 8; otherwise, a H pair will not permit a rigid body to pass through the prescribed positions.

4) Cylindric Pair: all those points whose several positions cannot be reached by means of a R, P, or H pair, can be reached by a C pair whose axis is the Burmester line, see Figure 9.

4.4. Identification of Cranks

In order to identify the cranks, we apply the conditions described in the previous section for each Burmester line pair; for instance, if the direction cosines and location of a Burmester line in the moving rigid body satisfy the conditions for a particular kinematic pair, say a R pair; and if the direction cosines and location of the corresponding Burmester line in the fixed rigid body satisfy the conditions for another kinematic pair, say a P pair, then we have a P - R crank.

Table VIII exhibits all the possible cranks containing R, P, H and C pairs.

TABLE VIII

BINARY CRANKS CONTAINING R, P, H AND C PAIRS

R - R	P - R	H - R	C - R
R - P	P - P	H - P	C - P
R - H	P - H	H - H	C - H
R - C	P - C	H - C	C - C

Once we have identified the crank, we need to specify the distance

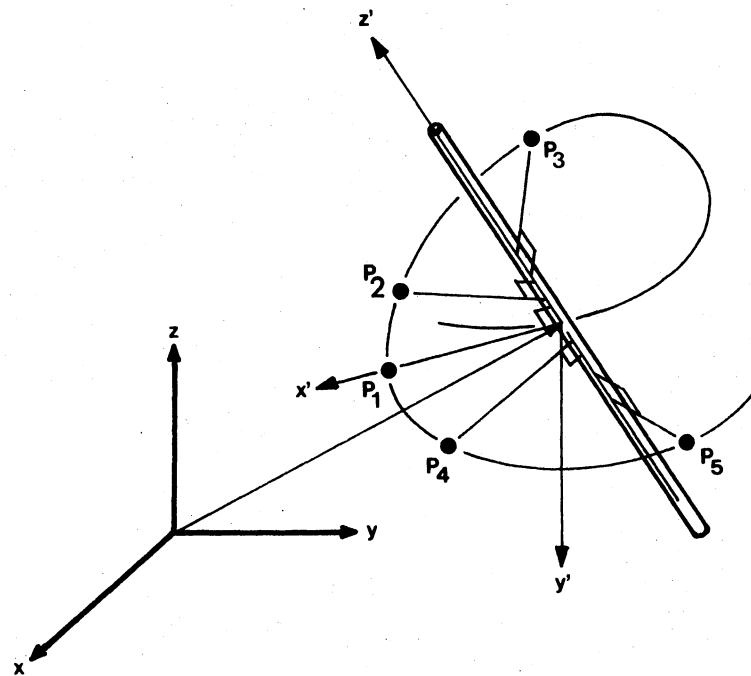


Figure 8. Points on a Helix

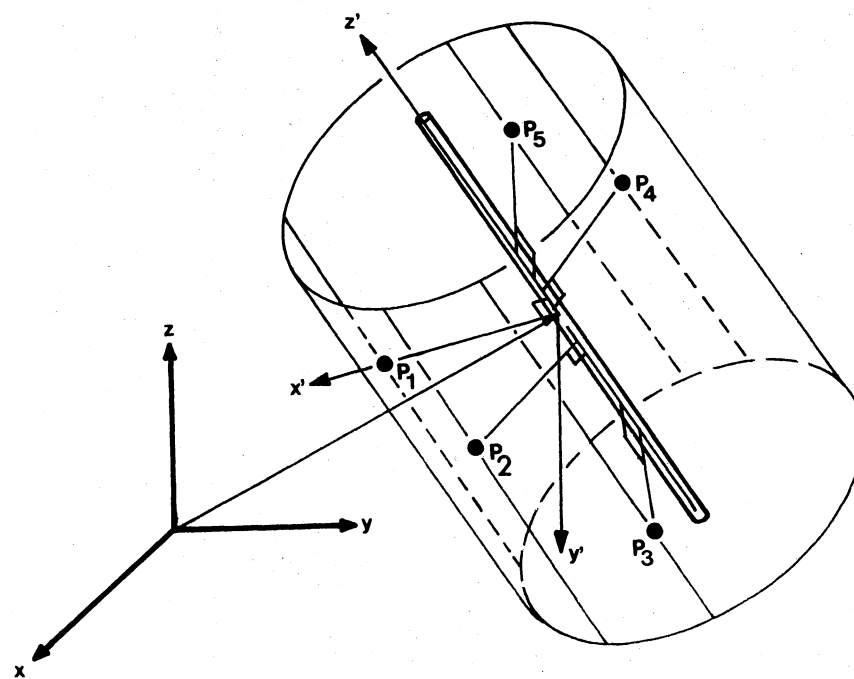


Figure 9. Points on a Cylinder

s from one of the kinematic pairs to the other one in the crank in study, as well as the twist angle Φ the axis of one of the kinematic pairs makes with the axis of the other one. The distance s is given by Equation (4.8). The twist angle Φ is

$$\Phi = \arccos (\ell_f \ell_m + m_f m_m + n_f n_m) \quad (4.10)$$

where

(ℓ_f, m_f, n_f) = direction cosines of a unit vector parallel to the Burmester line in the fixed rigid body, and

(ℓ_m, m_m, n_m) = direction cosines of a unit vector parallel to the Burmester line in the moving rigid body.

Once all the possible cranks have been identified (for five finitely separated positions of a rigid body), then we can use any two such cranks to synthesize a linkage that will displace a moving rigid body through its five finitely separated positions relative to a fixed rigid body. If at least one of the kinematic pairs of this linkage is a R, P, or H pair, then by grounding the pairs in the fixed rigid body, the synthesized linkage may become a mechanism with one degree of freedom.

CHAPTER V

SUMMARY AND CONCLUSIONS

The main objective of the present investigation was to derive an analytical expression which permits one to determine the Burmester lines for the problem of guiding a rigid body through its five finitely separated positions. In practice, the solution to this problem is accomplished by using a binary C - C crank. The analytical procedure involved in obtaining the Burmester lines is based on the application of the relationships for equivalent screw triangles formed by the binary crank and the screws describing the finitely separated positions. The four equivalent screw triangles formed by the binary crank and the screws yield a total of four equations relating the direction cosines and angles of rotation of the rigid body about the screws. Since there are only three independent relationships between direction cosines and angles of rotation, the set of four compatible equations yields two conditions forcing two third order determinants to vanish. Expansion of these determinants yields two cubic cones describing the space analogs of the center point curves in plane. The two cubic cones, in theory, will intersect in at most nine lines out of which there will be three screws and six Burmester lines. Analytically, we can eliminate the screws from the intersection of the two cubic cones and obtain a sixth degree polynomial whose real roots will provide the ratios of two of the three direction cosines for each of the six Burmester

lines. By introducing these ratios one by one in the equation of one of the cubic cones, we can determine the third direction cosine for each one of the Burmester lines. Once the direction cosines for each of the Burmester lines (in the moving and fixed frames of reference) have been found, the location of the Burmester lines in the moving rigid body and in the fixed rigid body can be determined.

Each Burmester line in the moving rigid body has a corresponding Burmester line in the fixed rigid body. This pair of Burmester lines constitutes a Burmester line pair. For design purposes each Burmester line pair may be replaced by a C - C crank. Furthermore, by restricting the rotation and/or translation of each of the C pairs of a C - C crank, we may obtain other types of cranks containing R, P, H or C pairs, such as the ones shown in Table VIII.

In summary, the contributions of the present investigation are:

- (1) For five finitely separated positions of a rigid body, a sixth degree polynomial whose real roots lead to the determination of the Burmester lines for C - C cranks is derived analytically.
- (2) The conditions under which a C - C crank may degenerate into cranks containing R, P, H or C pairs are given.

The derivation of the sixth degree polynomial involved an enormous amount of algebraic manipulation which, because of its complexity, was carried out using the available algebraic manipulation programs such as FORMAC and ALFRED.

The total amount of time required to reduce the seventh degree polynomial described by Equation (3.27) to the sixth degree polynomial described by Equation (3.29) exceeded more than 100 hours of CPU time

on an IBM 370-158, many times requiring as much as 1500 K-bytes of memory. There is still one more level of substitution for the complete simplification of each one of the coefficients of the sixth degree polynomial. We estimate that this task will require at least another 100 hours of CPU time on the same type of computer. When this final substitution is finished, we expect some of the coefficients of the sixth degree polynomial displayed in Table VII to be reduced as much as 70% of their actual size. We also predict that the term $(\tan^2 \theta_{13/2} + 1)$ will factor out completely from each one of the coefficients, thus simplifying these coefficients even more.

It is important to note that the determination of the direction cosines of the Burmester lines can be carried out by applying a numerical approach to determine the intersections of the two cubic cones given by Equations (3.19) and (3.20). Such a numerical approach, however, will provide along with the Burmester lines the three unwanted screws $\$_{12}$, $\$_{13}$, and $\$_{23}'$. In addition, initial estimates of some of the direction cosines are needed in order to proceed with their determination. The present investigation has developed an analytical approach that eliminates both of these problems. This is accomplished in the following steps: (1) selecting an appropriate frame of reference to describe all the screws, (2) applying relationships for the equivalent screw triangle and (3) eliminating analytically the common roots involving the direction cosines of screws $\$_{12}$, $\$_{13}$, and $\$_{23}'$. This final step yields the sixth degree polynomial whose real roots give the ratios of two of the three direction cosines of the Burmester lines in the moving frame of reference. The coefficients of this sixth degree polynomial (in its present form) may be used to perform theoretical studies in space, as

will be discussed shortly. However, numerical substitution in the coefficients of the sixth degree polynomial (3.29) is discouraged at the present, unless a very accurate computer is used. The coefficients of the sixth degree polynomial in their present form are susceptible to truncation and round-off errors. We expect that this problem will disappear once the final level of substitution is performed.

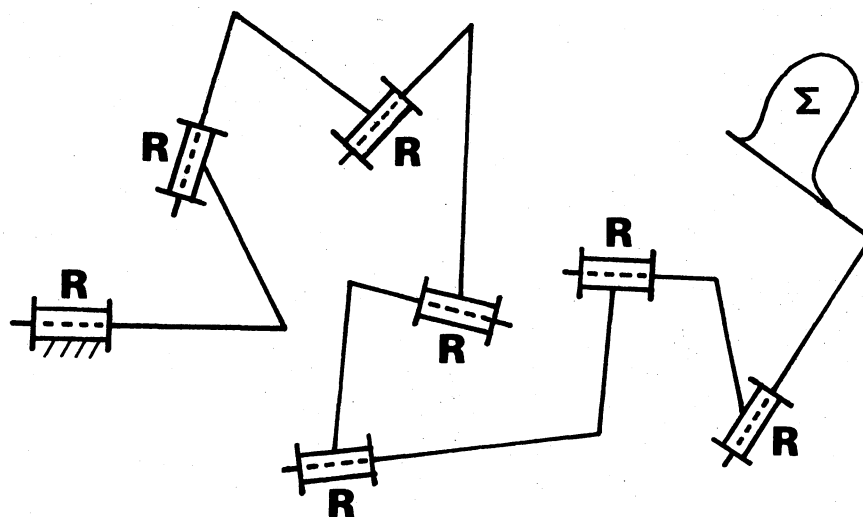
Direct application of the present investigation will permit a kinematician to undertake theoretical and applied research in Burmester theory. Some of the opportunities for applied research include:

- 1) Robots for handling radioactive or other hazardous materials.

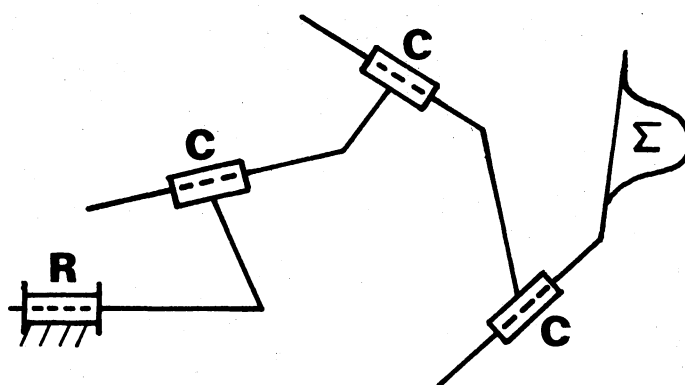
Most of the operations for handling radioactive or other hazardous materials require personal involvement. By properly programming the motion of the robot arm, and by using the sixth degree polynomial, the cranks can be positioned in such a way that it will actually perform the task without human intervention. The advantages are: (a) it is safer; (b) the operation can be performed faster; (c) many times the operation can be performed by a much simpler robot arm; i.e., with less number of cranks.

In many cases if a programmed motion is required, a robot arm such as the one shown in Figure 10a may be replaced by a simpler robot arm, such as the one shown in Figure 10b. The rigid body Σ in both robot arms has six degrees of freedom. The difference between the robot arms lies strictly in the number of components; i.e., the robot arm in Figure 10a contains seven R pairs and six cranks, while the one in Figure 10b contains one R pair, three C pairs and only three cranks. By selecting a series of five finitely separated positions of the rigid

body Σ , we can apply the sixth degree polynomial (3.29) and obtain the positions of the cranks that will displace the rigid body through its five prescribed positions.



(a)



(b)

Figure 10. (a) A RRRRRRR Open Loop Chain
(b) A RCCC Open Loop Chain

- 2) Robots for remote work, such as the ones in Mars and the moon.

All of the advantages of (1) above apply here. Furthermore, in many instances the robot arm is required to avoid obstacles, such as the legs of the landing unit, heavy rocks, etc. By selecting a series of five finitely separated positions and applying the results of this investigation, we can find the cranks which will pick up objects and at the same time avoid obstacles.

- 3) Airplane landing gears.

By optimizing the RCCC open loop chain in figure 10b, we can probably find a new design for an airplane landing gear. This landing gear would function as the present landing gears in its landing mode, but once the craft is airborne, it could be folded efficiently in a more compact form.

Other applications of the results of this investigation are only limited to the designer's imagination. We expect new designs to appear in the areas of prosthetics, engine transmissions, optical equipment, textile machinery, packaging machinery, etc. in the near future.

The results of the present investigation also permit one to undertake a variety of research problems in three-dimensional kinematics. Some of these problems are briefly presented below:

- 1) Synthesis of three and four link mechanisms in space.

By using the results of this investigation, provided a very accurate computer is available, it is possible for one to synthesize mechanisms which will displace a rigid body through its five prescribed finitely separated positions.

In some design situations; however, one is required to design a mechanism which will displace a rigid body through its five prescribed

finitely separated positions, and at the same time prescribe the location and orientation of the fixed pivots. In other words, the Burmester lines in the fixed rigid body, as well as the five finitely separated positions of the rigid body, are given. This type of design situation is completely unexplored at the present. Now that an analytical approach that leads to the determination of the Burmester lines is available, it is expected that in the near future procedures to tackle the design problem of specified Burmester lines and positions of a rigid body will be developed.

In many cases a new improved design for an already existent mechanism is desired. By applying the results of this investigation, it is possible, in general, to obtain other cranks which, when used in a mechanism, will perform the same operations as the original mechanism, and at the same time may have an improved transmission angle or better dimensions.

Since Burmester theory deals strictly with relative motions, it is possible for one to apply the results of this investigation for the synthesis of mechanisms with more than four links. It is hoped that the derivation of the sixth degree polynomial whose real roots lead to the determination of the Burmester lines, will provide the user with a better kinematic insight to his particular design problem.

2) Properties of Burmester lines.

The five positions of a rigid body, Σ , moving relative to another rigid body, Σ' , determine the number of Burmester lines which displace Σ relative to Σ' . The Burmester lines are defined by the direction cosines of unit vectors parallel to the Burmester lines and a point on the Burmester line. It is reasonable to expect that for

different arrangements of the positions of a rigid body Σ , the number and location of the Burmester lines may vary.

Studies similar to the ones performed for plane kinematics, as were described in Table I, may be undertaken in order to extend some of the properties existent for planar motion into spherical and spatial motion. In space, a rigid body can be arranged in a greater number of different ways due to the third dimension. This fact permits one to study a set of properties that are unique to spatial motion. The research opportunities for this particular aspect of Burmester theory are unlimited.

3) Special cases of the sixth degree polynomial (3.29).

A research study by itself could be undertaken just to investigate the sixth degree polynomial whose real roots give the ratios of the Burmester lines in the moving rigid body. For instance, one could find under what conditions some of the coefficients vanish, and then relate the findings to physically attainable mechanisms. This type of study would give a better insight on the properties to be studied in (2) above.

4) Study of other cranks containing S, C, R, P or H pairs.

Our study was limited to the C - C cranks and all its variations with R, P, H or C pairs, as presented in Table VIII. There is, however, another very useful set of binary cranks which contains an S pair. In some instances, these cranks would permit us to design a mechanism for as many as eight positions of a rigid body. It is hoped that the present research will encourage other investigators to extend space Burmester theory to a new theory which includes more than five positions of a rigid body.

5) Extension of Burmester theory for diads to triads.

This type of study would be very useful in case a large number of

rigid body positions is desired. This would lead to the design of mechanisms with more than four links.

Since the study of triads in planar kinematics is fairly limited, we are left with a poor foundation and feel for what can be expected in space. Thus, it is anticipated that this particular problem will be the most difficult.

In view of the several areas of future research presented above, the outcome of the present investigation appears to be very limited. Yet, at the same time, we want to emphasize that this is the step which will permit kinematicians to explore new areas of applications and theory in spatial kinematics.

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APPENDIX A

HOW TO FIND THE DIRECTION COSINES OF A LINE WHICH IS PERPENDICULAR TO TWO INTERSECTING LINES

Let us define a line L_g with direction cosines (ℓ_g, m_g, n_g) to be perpendicular to lines L_e and L_f .

Line L_e is of the form

$$\frac{x - a_e}{\ell_e} = \frac{y - b_e}{m_e} = \frac{z - c_e}{n_e}, \quad (a.1)$$

and line L_f is of the form

$$\frac{x - a_f}{\ell_f} = \frac{y - b_f}{m_f} = \frac{z - c_f}{n_f}. \quad (a.2)$$

The condition for line L_g to be perpendicular to line L_e is:

$$\ell_g \ell_e + m_g m_e + n_g n_e = 0. \quad (a.3)$$

The condition for line L_g to be perpendicular to line L_f is:

$$\ell_g \ell_f + m_g m_f + n_g n_f = 0. \quad (a.4)$$

The direction cosines of line L_g must satisfy the following condition:

$$\ell_g^2 + m_g^2 + n_g^2 = 1. \quad (a.5)$$

We can now find (ℓ_g, m_g, n_g) from Equations (a.3 - a.5). The following cases may arise (in all cases $\text{sgn} = +1$ or -1):

Case 1.- If $\ell_e m_f - \ell_f m_e \neq 0$, then

$$n_g = \text{sgn} / \sqrt{R_a^2 + R_b^2 + 1}$$

where

$$R_a = \frac{n_{fe}^m - n_{ef}^m}{l_{ef}^m - l_{fe}^m}$$

$$R_b = \frac{n_{fe}^l - n_{ef}^l}{l_{fe}^m - l_{ef}^m}$$

and

$$l_g = R_a n_g$$

$$m_g = R_b n_g$$

Case 2.- If $l_{ef}^n - l_{fe}^n \neq 0$, then

$$m_g = \text{sgn} / \sqrt{R_a^2 + R_b^2 + 1}$$

where

$$R_a = \frac{m_{fe}^n - m_{ef}^n}{l_{ef}^n - l_{fe}^n}$$

$$R_b = \frac{m_{fe}^l - m_{ef}^l}{l_{fe}^n - l_{ef}^n}$$

and

$$l_g = R_a m_g$$

$$n_g = R_b m_g$$

Case 3.- If $m_{fe}^n - m_{ef}^n \neq 0$, then

$$l_g = \text{sgn} / \sqrt{R_a^2 + R_b^2 + 1}$$

where

$$R_a = \frac{l_{fe}^n - l_{ef}^n}{m_{ef}^n - m_{fe}^n}$$

$$R_b = \frac{l_{fe}^m - l_{ef}^m}{m_{fe}^n - m_{ef}^n}$$

and

$$m_g = R_a l_g$$

$$n_g = R_b l_g$$

APPENDIX B

DIRECTION COSINES OF THE BURMESTER LINES IN THE FIXED FRAME OF REFERENCE

The real roots of the sixth degree polynomial, Equation (3.29), give the direction cosines (M_x, M_y, M_z) of the Burmester lines in the moving rigid body. Thus, a_i , $i = 1, \dots, 5$ in Equations (3.13) and (3.14) are completely defined. Rewriting these equations we have

$$a_1 A + a_2 = 0 \quad (b.1)$$

$$a_3 A + a_5 B + a_4 = 0 \quad (b.2)$$

where

$$A = \frac{F_x}{F_y}$$

$$B = \frac{F_z}{F_y}$$

From Equation (b.1) we find that

$$A = -\frac{a_2}{a_1} \quad ; \quad (b.3)$$

then Equation (b.2) yields

$$B = \frac{a_2 a_3 - a_1 a_4}{a_1 a_5} \quad (b.4)$$

We know that the direction cosines (F_x, F_y, F_z) obey the following condition:

$$F_x^2 + F_y^2 + F_z^2 = 1 \quad ; \quad (b.5)$$

thus

$$F_y = 1 / \sqrt{A^2 + B^2 + 1}$$

and

$$F_x = A F_y$$

$$F_z = B F_y$$

We find that there is only one Burmester line in the fixed rigid body corresponding to each one of the Burmester lines in the moving rigid body. Each set of moving and fixed Burmester lines forms a Burmester line pair (which is the space analog of the Burmester point pair in plane kinematics).

APPENDIX C

LOCATION OF THE BURMESTER LINES CORRESPONDING TO THE MOVING FRAME OF REFERENCE

In Chapter III we found that the intersections of the two screw cones (Figure 4) give us the condition for a line embedded in the moving rigid body, in its five finitely separated positions, to be at a constant inclination relative to another line embedded in the fixed rigid body. This condition, however, is not sufficient for the complete description of a Burmester line. We also need to satisfy the condition that the line in the moving rigid body remains at a constant distance from some line in the fixed rigid body (see Figure 11 for nomenclature).

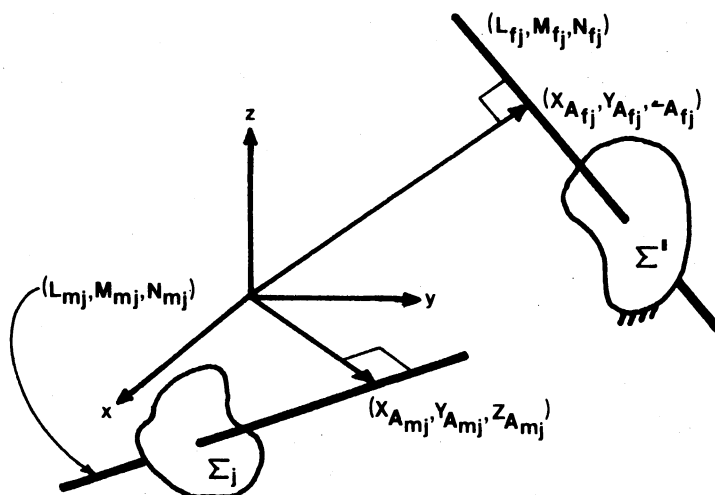


Figure 11. Nomenclature Used in the Location of the Burmester Lines

The conditions for constant inclination (or twist) and distance between two Burmester lines are given by [62]

$$\cos \Phi_1 = \cos \Phi_2 = \cos \Phi_3 = \cos \Phi_4 = \cos \Phi_5, \quad ,$$

and

$$s_1 \sin \Phi_1 = s_2 \sin \Phi_2 = s_3 \sin \Phi_3 = s_4 \sin \Phi_4 = s_5 \sin \Phi_5$$

where s_i and Φ_i , $i = 1, \dots, 5$ are the distance and angle between the lines in the moving and fixed frames of reference.

The constant moment condition gives us

$$s_i \sin \Phi_i = L_{mj} G_1 + M_{mj} G_2 + N_{mj} G_3 + L_{fj} G_1' + M_{fj} G_2' + N_{fj} G_3' \quad (c.1)$$

where (G_1, G_2, G_3) is the cross product of the unit vector parallel to the Burmester line in the fixed frame of reference with direction cosines

(L_{fj}, M_{fj}, N_{fj}) and the vector from the origin to the point $(x_{Afj}, y_{Afj}, z_{Afj})$ on this Burmester line; and (G_1', G_2', G_3') is the cross product of the unit vector parallel to the Burmester line in the moving frame of reference with direction cosines (L_{mj}, M_{mj}, N_{mj}) and the vector from the origin to the point $(x_{Amj}, y_{Amj}, z_{Amj})$ on the moving Burmester line.

In order to locate the Burmester line corresponding to the moving rigid body, proceed as follows:¹⁹

- 1) Redefine the direction cosines (L_{mj}, M_{mj}, N_{mj}) of the Burmester line in study as (L_i, M_i, N_i) , $i = 1, \dots, 5$.
- 2) Define (L_i, M_i, N_i) as

$$\begin{bmatrix} L_i \\ M_i \\ N_i \end{bmatrix} = \begin{bmatrix} (a_{xi} + 1) & b_{xi} & c_{xi} \\ a_{yi} & (b_{yi} + 1) & c_{yi} \\ a_{zi} & b_{zi} & (c_{zi} + 1) \end{bmatrix} \begin{bmatrix} L_1 \\ M_1 \\ N_1 \end{bmatrix}$$

¹⁹ There is an alternate procedure given in reference [104]; that procedure, however, is not completely general, since it does not include the case when either $\$14$ and/or $\$15$ and/or a Burmester line is perpendicular to $\$12$. The procedure here is similar to the one given in reference [62].

where

$$\begin{aligned}
 a_{xi} &= (u_{1i}^2 - 1)(1 - \cos \theta_i) \\
 b_{xi} &= u_{1i}u_{2i}(1 - \cos \theta_i) - u_{3i} \sin \theta_i \\
 c_{xi} &= u_{1i}u_{3i}(1 - \cos \theta_i) + u_{2i} \sin \theta_i \\
 a_{yi} &= u_{1i}u_{2i}(1 - \cos \theta_i) + u_{3i} \sin \theta_i \\
 b_{yi} &= (u_{2i}^2 - 1)(1 - \cos \theta_i) \\
 c_{yi} &= u_{2i}u_{3i}(1 - \cos \theta_i) - u_{1i} \sin \theta_i \\
 a_{zi} &= u_{1i}u_{3i}(1 - \cos \theta_i) - u_{2i} \sin \theta_i \\
 b_{zi} &= u_{2i}u_{3i}(1 - \cos \theta_i) + u_{1i} \sin \theta_i \\
 c_{zi} &= (u_{3i}^2 - 1)(1 - \cos \theta_i)
 \end{aligned}$$

for $i = 2, \dots, 5$; here (u_{1i}, u_{2i}, u_{3i}) and θ_i are the direction cosines and rotation angles corresponding to screws $\$_{1i}$, $i = 2, \dots, 5$.

3) Define F_{1i} , F_{2i} and F_{3i} as

$$\begin{aligned}
 F_{1i} &= L_i - L_1 \\
 F_{2i} &= M_i - M_1 \\
 F_{3i} &= N_i - N_1
 \end{aligned}$$

for $i = 2, \dots, 5$.

4) Rewrite Equation (c.1) in the following form:

$$G_1 F_{1i} + G_2 F_{2i} + G_3 F_{3i} = -E_i \quad (c.2)$$

where

$$\begin{aligned}
 E_i &= x_{Amj} K_{1i} + y_{Amj} K_{2i} + z_{Amj} K_{3i} + K_{4i} ; \\
 K_{1i} &= L_{fj}(N_i a_{yi} - M_i a_{zi}) + M_{fj}(L_i a_{zi} - N_i a_{xi} - F_{3i}) + N_{fj}(M_i a_{xi} - L_i a_{yi} + F_{3i}) \\
 K_{2i} &= L_{fj}(N_i b_{yi} - M_i b_{zi} + F_{3i}) + M_{fj}(L_i b_{zi} - N_i b_{xi}) + N_{fj}(M_i b_{xi} - L_i b_{yi} - F_{1i}) \\
 K_{3i} &= L_{fj}(N_i c_{yi} - M_i c_{zi} - F_{2i}) + M_{fj}(L_i c_{zi} - N_i c_{xi} + F_{1i}) + N_{fj}(M_i c_{xi} - L_i c_{yi}) \\
 K_{4i} &= L_{fj}(N_i d_{yi} - M_i d_{zi}) + M_{fj}(L_i d_{zi} - N_i d_{xi}) + N_{fj}(M_i d_{xi} - L_i d_{yi})
 \end{aligned}$$

and

$$d_{xi} = d_i u_{1i} - a_i a_{xi} - b_i b_{xi} - c_i c_{xi}$$

$$d_{yi} = d_i u_{2i} - a_i a_{yi} - b_i b_{yi} - c_i c_{yi}$$

$$d_{zi} = d_i u_{3i} - a_i a_{zi} - b_i b_{zi} - c_i c_{zi}$$

for $i = 2, \dots, 5$; here (a_i, b_i, c_i) are the coordinates of a point on $\$_{1i}$,

$i = 2, \dots, 5$, and d_i are the translations along screws $\$_{1i}$, $i = 2, \dots, 5$.

5) By forcing the line from the origin to the Burmester line corresponding to the fixed frame of reference to be perpendicular to the Burmester line, we obtain

$$G_1 L_{fj} + G_2 M_{fj} + G_3 N_{fj} = 0 \quad (c.3)$$

6) Equation (c.2) for $i = 2, \dots, 5$ and Equation (c.3) permit us to find

G_1 , G_2 , and G_3 . For this purpose, let us define

$$\begin{vmatrix} L_{fj} & M_{fj} & N_{fj} & 0 \\ F_{12} & F_{22} & F_{32} & -E_2 \\ F_{13} & F_{23} & F_{33} & -E_3 \\ F_{14} & F_{24} & F_{34} & -E_4 \end{vmatrix} = 0 \quad (c.4)$$

and

$$\begin{vmatrix} L_{fj} & M_{fj} & N_{fj} & 0 \\ F_{12} & F_{22} & F_{32} & -E_2 \\ F_{13} & F_{23} & F_{33} & -E_3 \\ F_{15} & F_{25} & F_{35} & -E_5 \end{vmatrix} = 0 \quad (c.5)$$

Now, Equations (c.4) and (c.5) may be simplified into

$$x_{Amj} H_1 + y_{Amj} H_2 + z_{Amj} H_3 + H_4 = 0 \quad (c.6)$$

$$x_{Amj} H_5 + y_{Amj} H_6 + z_{Amj} H_7 + H_8 = 0 \quad (c.7)$$

where

$$H_i = K_{1i} D_a + K_{3i} D_b$$

$$H_{i+4} = K_{1i} D_c + K_{4i} D_b$$

for $i = 1, \dots, 4$ and

$$D_a = \begin{vmatrix} L_{fj} & M_{fj} & N_{fj} \\ F_{13} & F_{23} & F_{33} \\ F_{14} & F_{24} & F_{34} \end{vmatrix} ; \quad D_b = \begin{vmatrix} L_{fj} & M_{fj} & N_{fj} \\ F_{12} & F_{22} & F_{32} \\ F_{13} & F_{23} & F_{33} \end{vmatrix} ;$$

$$D_c = \begin{vmatrix} L_{fj} & M_{fj} & N_{fj} \\ F_{13} & F_{23} & F_{33} \\ F_{15} & F_{25} & F_{35} \end{vmatrix} .$$

Now, Equations (c.6) and (c.7) together with

$$x_{Amj} L_{mj} + y_{Amj} M_{mj} + z_{Amj} N_{mj} = 0 \quad (c.8)$$

give us three equations and three unknowns in $(x_{Amj}, y_{Amj}, z_{Amj})$; thus

defining

$$D_{mj} = \begin{vmatrix} H_1 & H_2 & H_3 \\ H_5 & H_6 & H_7 \\ L_{mj} & M_{mj} & N_{mj} \end{vmatrix} ; \quad D_{xm} = \begin{vmatrix} H_4 & H_2 & H_3 \\ H_8 & H_6 & H_7 \\ 0 & M_{mj} & N_{mj} \end{vmatrix} ;$$

$$D_{ym} = \begin{vmatrix} H_1 & H_4 & H_3 \\ H_5 & H_8 & H_7 \\ L_{mj} & 0 & N_{mj} \end{vmatrix} ; \quad \text{and} \quad D_{zm} = \begin{vmatrix} H_1 & H_2 & H_4 \\ H_5 & H_6 & H_8 \\ L_{mj} & M_{mj} & 0 \end{vmatrix}$$

we obtain

$$x_{Amj} = \frac{D_{xm}}{D_{mj}} ; \quad y_{Amj} = \frac{D_{ym}}{D_{mj}} ; \quad \text{and} \quad z_{Amj} = \frac{D_{zm}}{D_{mj}} .$$

To find the coordinates of a point on each of the Burmester lines in the moving frame of reference, repeat the above procedure for each $j = 1, \dots, 2k$, where $k = 0, 1, 2$, or 3 .

APPENDIX D

LOCATION OF THE BURMESTER LINES CORRESPONDING TO THE FIXED FRAME OF REFERENCE

To find²⁰ the coordinates of a point on the Burmester axis embedded in the fixed frame of reference, we use the following equations:

$$G_1 F_{12} + G_2 F_{22} + G_3 F_{32} = -E_2 \quad (d.1)$$

$$G_1 F_{13} + G_2 F_{23} + G_3 F_{33} = -E_3 \quad (d.2)$$

Equations (d.1) and (d.2), together with Equation (c.3) give us three equations and three unknowns in G_1 , G_2 , and G_3 ; thus defining

$$D_G = \begin{vmatrix} F_{12} & F_{22} & F_{32} \\ F_{13} & F_{23} & F_{33} \\ L_{fj} & M_{fj} & N_{fj} \end{vmatrix} ; \quad D_{G1} = \begin{vmatrix} -E_2 & F_{22} & F_{32} \\ -E_3 & F_{23} & F_{33} \\ 0 & M_{fj} & N_{fj} \end{vmatrix} ;$$

$$D_{G2} = \begin{vmatrix} F_{12} & -E_2 & F_{32} \\ F_{13} & -E_3 & F_{33} \\ L_{fj} & 0 & N_{fj} \end{vmatrix} ; \text{ and } \quad D_{G3} = \begin{vmatrix} F_{12} & F_{22} & -E_2 \\ F_{13} & F_{23} & -E_3 \\ L_{fj} & M_{fj} & 0 \end{vmatrix}$$

we obtain

$$G_1 = \frac{D_{G1}}{D_G} ; \quad G_2 = \frac{D_{G2}}{D_G} ; \quad \text{and} \quad G_3 = \frac{D_{G3}}{D_G} .$$

Now, since (G_1, G_2, G_3) is the cross product of the unit vector with direction cosines (L_{fj}, M_{fj}, N_{fj}) and the vector from the origin to the point $(x_{Afj}, y_{Afj}, z_{Afj})$ on the Burmester line in the fixed frame of reference, then the following relationships are true

²⁰ See footnote number 19 .

$$y_{Afj} N_{fj} - z_{Afj} M_{fj} = G_1 \quad , \quad (d.3)$$

$$z_{Afj} L_{fj} - x_{Afj} N_{fj} = G_2 \quad , \quad (d.4)$$

$$x_{Afj} M_{fj} - y_{Afj} L_{fj} = G_3 \quad . \quad (d.5)$$

Here, we have three equations and three unknowns in $(x_{Afj}, y_{Afj}, z_{Afj})$,

thus by defining

$$D_{fj} = \begin{bmatrix} 0 & N_{fj} & -M_{fj} \\ -N_{fj} & 0 & L_{fj} \\ M_{fj} & -L_{fj} & 0 \end{bmatrix} ; \quad D_{xf} = \begin{bmatrix} G_1 & N_{fj} & -M_{fj} \\ G_2 & 0 & L_{fj} \\ G_3 & -L_{fj} & 0 \end{bmatrix}$$

$$D_{yf} = \begin{bmatrix} 0 & G_1 & -M_{fj} \\ -N_{fj} & G_2 & L_{fj} \\ M_{fj} & G_3 & 0 \end{bmatrix} ; \text{ and } D_{zf} = \begin{bmatrix} 0 & N_{fj} & G_1 \\ -N_{fj} & 0 & G_2 \\ M_{fj} & -L_{fj} & G_3 \end{bmatrix}$$

we obtain

$$x_{Afj} = \frac{D_{xf}}{D_{fj}} ; \quad y_{Afj} = \frac{D_{yf}}{D_{fj}} ; \quad \text{and} \quad z_{Afj} = \frac{D_{zf}}{D_{fj}} .$$

To find the coordinates of a point on each of the Burmester lines in the fixed frame of reference, repeat the above procedure for each $j = 1, \dots, 2k$, where $k = 0, 1, 2$, or 3 .

APPENDIX E

COMPUTER PROGRAM

This appendix presents the computer program used to determine the real roots of Equation (3.29). The numerical values of the screws used in this program are the same as the ones given by Roth in Reference [62].

The output of the program presents:

- 1) The direction cosines of the Burmester lines as given by Roth,
- 2) The direction cosines of the Burmester lines in the transformed coordinate system,
- 3) The values of Equations (3.19), (3.20), and (3.26) with the direction cosines of the Burmester lines in the transformed coordinate system,
- 4) The values of T_{ij} , $i = 0, \dots, 6$, $j = 0, \dots, 3$,
- 5) The coefficients of the sixth degree polynomial (3.29), and
- 6) The real and imaginary roots of the sixth degree polynomial.

The real roots of Equation (3.29) are the ratios of the direction cosines l/m corresponding to the Burmester lines.


```

READ(5,101)U12,V12,U13,V13
READ(5,101)U14,V14,U15,V15
W12= DSQRT(1.-U12**2-V12**2)
W13= DSQRT(1.-U13**2-V13**2)
W14= DSQRT(1.-U14**2-V14**2)
W15=DSQRT(1.-U15**2-V15**2)
READ(5,100)A12,B12,C12,A13,B13,C13
READ(5,101)T2,T3,T4,T5
I=2
WRITE(6,103)I,U12,V12,W12
WRITE(6,104)I,T2
I=3
WRITE(6,103)I,U13,V13,W13
WRITE(6,104)I,T3
I=4
WRITE(6,103)I,U14,V14,W14
WRITE(6,104)I,T4
I=5
WRITE(6,103)I,U15,V15,W15
WRITE(6,104)I,T5
I=2
WRITE(6,105)I,A12,B12,C12
I=3
WRITE(6,105)I,A13,B13,C13
R1=U12*U13+V12*V13+W12*W13
R2=U12*(A12-A13)+V12*(B12-B13)+W12*(C12-C13)
R3=U13*(A12-A13)+V13*(B12-B13)+W13*(C12-C13)
R4=(R1*R3-R2)/(1.-R1**2)
R5=R1*R4+R3
P1=A12+U12*R4
P2=B12+V12*R4
P3=C12+W12*R4
P11=A13+U13*R5
P12=B13+V13*R5
P13=C13+W13*R5
S23=DSQRT((P11-P1)**2+(P12-P2)**2+(P13-P3)**2)
LA=(P11-P1)/S23
MA=(P12-P2)/S23
NA=(P13-P3)/S23
RE=(NA*V12-MA*W12)/(MA*U12-LA*V12)
RF=(U12*NA-W12*LA)/(V12*LA-U12*MA)
NB=-1./DSQRT(1.+RE**2+RF**2)
LB=RE*NB
MB=RF*NB
WRITE(6,106)
WRITE(6,107)LB,NB,MB
WRITE(6,108)LA,MA,NA
WRITE(6,109)U12,V12,W12
U12=0.
V12=0.
W12=1.
U24=U14*LB+V14*MB+W14*NB
V24=U14*LA+V14*MA+W14*NA
W24=U14*U12+V14*V12+W14*W12
U25=U15*LB+V15*MB+W15*NB
V25=U15*LA+V15*MA+W15*NA
W25=U15*U12+V15*V12+W15*W12
U23=U15*LB+V15*MB+W15*NB

```

```

V23=0.
W23=U13*U12+V13*V12+W13*W12
I=2
WRITE(6,102)I,U12,V12,W12
I=3
WRITE(6,102)I,U23,V23,W23
I=4
WRITE(6,102)I,U24,V24,W24
I=5
WRITE(6,102)I,U25,V25,W25
D=3.1415/180.
N1=DTAN(T2*D/2.)
N2=DTAN(T3*D/2.)
N3=DTAN(T4*D/2.)
N4=DTAN(T5*D/2.)
RETURN
END

```

```

SUBROUTINE LYNDA(K)
IMPLICIT REAL*8 (A-Z)
DIMENSION C(15),E(18),K(34)
COMMON/INPUT/U23,W23,U24,V24,W24,U25,V25,W25,N2,N3,N4
C(1)=N3*(1-W24**2)
C(2)=N3*(1-V24**2)
C(3)=N3*(1-U24**2)
C(5)=N3*U24*W24
C(6)=N3*V24*W24
C(7)=N3*U24*V24
C(9)=N4*(1-W25**2)
C(10)=N4*(1-V25**2)
C(11)=N4*(1-U25**2)
C(13)=N4*U25*W25
C(14)=N4*V25*W25
C(15)=N4*U25*V25

```

DEFINITION OF E'S FROM TABLE VI

```

F(1)=C(1)/U23
E(2)=C(5)-C(1)*W23/U23+V24
E(3)=W23*U24-W23*C(6)-U23*C(7)-U23*W24
E(4)=C(5)-C(1)*W23/U23-V24
E(5)=C(1)*W23**2/U23-2*C(5)*W23+U23*C(3)
E(6)=2*C(6)
E(7)=U23*C(7)-U23*W24+W23*C(6)+W23*U24
E(8)=-C(6)-U24
E(9)=U23*C(2)
E(10)=C(9)/U23
E(11)=C(13)-C(9)*W23/U23+V25
E(12)=W23*U25-W23*C(14)-U23*C(15)-U23*W25
E(13)=C(13)-C(9)*W23/U23-V25
E(14)=C(9)*W23**2/U23-2*C(13)*W23+U23*C(11)
E(15)=2*C(14)
E(16)=U23*C(15)-U23*W25+W23*C(14)+W23*U25
E(17)=-C(14)-U25
E(18)=U23*C(10)
K(1)=N2*(E(6)+E(8))+E(1)
K(2)=N2*E(2)
K(3)=N2*E(3)+E(4)

```



```

K(4)=N2*E(5)
K(6)=N2*(E(3)-E(7))+2.*(E(4)+V24)
K(7)=N2*E(7)-E(2)
K(8)=N2*(E(9)-E(1))+E(6)
K(9)=-N2*E(4)+E(3)
K(10)=E(5)
K(13)=E(7)
K(14)=N2*E(8)+E(9)
K(15)=N2*(E(15)+E(17))+E(10)
K(16)=N2*E(11)
K(17)=N2*E(12)+E(13)
K(18)=N2*E(14)
K(20)=N2*(E(12)-E(16))+2.*(E(13)+V25)
K(21)=N2*E(16)-E(11)
K(22)=N2*(E(18)-E(10))+E(15)
K(23)=-N2*E(13)+E(12)
K(24)=E(14)
K(27)=E(16)
K(28)=N2*E(17)+E(18)
K(5)=K(18)-K(4)
K(11)=K(14)-K(10)
K(12)=K(9)-K(13)
K(19)=K(22)-K(18)
K(25)=K(26)-K(24)
K(26)=K(23)-K(27)
RETURN
END

```

```

SUBROUTINE MARY(TT,A)
IMPLICIT REAL*8 (A-H,K-Z)
DIMENSION TT(28),A(7)
COMMON/NN1/N1

```

COEFFICIENTS OF THE SIXTH DEGREE POLYNOMIAL (3,29)

```

100 FORMAT(2X,'G',I1,' : ',F19.8)
101 FORMAT(//,2X,'THE COEFFICIENTS OF THE SIXTH DEGREE POLYNOMIAL ARE:
5',/)
J=0
WRITE(6,101)
DO 10 I=1,7
A(I)=TT(J+I)*N1**3+TT(J+I+1)*N1**2+TT(J+I+2)*N1+TT(J+I+3)
J=J+3
JJ=-I-1
10 WRITE(6,100) JJ,A(I)
RETURN
END

```

```

SUBROUTINE BETH(K,TT)
IMPLICIT REAL*8 (A-H,J-Z)
DIMENSION K(34),TT(28)
K1=K(1)
K2=K(2)
K3=K(3)
K4=K(4)
K5=K(5)
K6=K(6)
K7=K(7)

```

```

K8=K(8)
K9=K(9)
K10=K(10)
K11=K(11)
K12=K(12)
K13=K(13)
K14=K(14)
K15=K(15)
K16=K(16)
K17=K(17)
K18=K(18)
K19=K(19)
K20=K(20)
K21=K(21)
K22=K(22)
K23=K(23)
K24=K(24)
K25=K(25)
K26=K(26)
K27=K(27)
K28=K(28)

```

DEFINITION OF THE COEFFICIENTS OF THE SIXTH DEGREE POLYNOMIAL FROM TABLE VII

```

I03 = -K1*K3*K17*K23+K1*K4*K17*K23-K3*K9*K15*K17+K15*K23*K3
***2
T02 = -K1*K3*K17*K24-K1*K3*K18*K23-K1*K4*K17*K23+2*K1*K5*
K17*K18+K1*K10*K17**2-K2*K3*K17*K23+K2*K4*K17*K23+2*K1*K5*
K3*K9*K15*K18-K3*K9*K16*K17-K3*K10*K15*K17-K4*K7*K15*K18-K4*K7*
K16*K23*K5**2
T01 = -K1*K3*K18*K24-K1*K4*K17*K24-K1*K4*K18*K23+K1*K9*K18
K5**2+2*K1*K10*K17*K18-K2*K5*K17*K24-K2*K3*K18*K23+2*
K2*K9*K17*K18+K2*K10*K17**2+2*K1*K4*K15*K24+2*K1*K4*K16*K23-K3*K9
K16*K18-K3*K10*K15*K18-K3*K17*K16*K17-K4*K7*K15*K18-K4*K7*
K16*K23*K5**2
T00 = -K1*K4*K15*K24+K1*K10*K18**2-K2*K5*K18*K24-K2*K4*K17
K5*K23-K2*K4*K16*K23+K2*K9*K16**2+2*K2*K10*K17*K15+2*K3*K4*K15*
K17-K3*K10*K16*K18-K4*K9*K16*K18-K4*K10*K15*K18-K4*K16*K17-
K24*K4**2+K16*K23*K4**2
T13 = K1*K3*K17*K24-K1*K3*K17*K28+K1*K3*K18*K23-K1*K5*K22*
K23+K1*K4*K17*K23-K1*K8*K17*K23-K1*K9*K15*K23-2*K1*K9*K17*
K16+2*K1*K9*K17*K22-K1*K10*K17**2+K1*K14*K17**2+K2*K5*K17*
K23-K2*K9*K17*K23+2*K3*K4*K15*K23+2*K3*K9*K15*K23+K3*K9*K15*
K18-K3*K9*K15*K15
K22+K3*K9*K16*K17+K3*K10*K15*N1/-K3*K14*K15*K17+K4*K9*K15*
K17-K4*K9*K15*K17-K15*K24*K3**2+K15*K28*K3**2-K16*K23*K3**2+
K15**2*K23**2
T12 = 2*K1*K2*K23**2-K1*K3*K17*K23+K1*K3*K17*K27+2*K1*K3*
K18*K24-K1*K3*K18*K28-K1*K3*K20*K23-K1*K5*K22*K24+2*K1*K4*
K17*K24-K1*K4*K17*K28+2*K1*K4*K18*K23-K1*K4*K22*K23-K1*K6*
K17*K23-K1*K6*K17*K28+2*K1*K9*K15*K24-2*K1*K9*K16*K23+2*K1*
K9*K17*K23+2*K1*K9*K17*K28+2*K1*K9*K18*K23+2*K1*K10*K17*
K22-K1*K13*K17*K22+2*K1*K14*K17*K18+2*K1*K10*K17*K18+2*K1*
K10*K17*K18+2*K1*K10*K17*K22-K1*K13*K17*K22+2*K1*K14*K17*
K18+2*K1*K10*K17*K18+2*K1*K10*K17*K22-K1*K13*K17*K22+2*K1*
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K22+2*K1*K14*K17*K18+2*K1*K10*K17*K18+2*K1*K10*K17*K22-
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K3*K84*K15*K23+2*K1*K3*K84*K15*K24+2*K1*K3*K85*K15*K23+2*K1*
K3*K85*K15*K24+2*K1*K3*K85*K15*K23+2*K1*K3*K85*K15*K24+2*K1*
K3*K85*K15*K23+2*K1*K3*K85*K15*K24+2*K1*K3*K86*K15*K23+2*K1*
K3*K86*K15*K24+2*K1*K3*K86*K15*K23+2*K1*K3*K86*K15*K24+2*K1*
K3*K86*K15*K23+2*K1*K3*K86*K15*K24+2*K1*K3*K87*K15*K23+2*K1*
K3*K87*K15*K24+2*K1*K3*K87*K15*K23+2*K1*K3*K87*K15*K24+2*K1*
K3*K87*K15*K23+2*K1*K3*K87*K15*K24+2*K1*K3*K88*K15*K23+2*K1*
K3*K88*K15*K24+2*K1*K3*K88*K15*K23+2*K1*K3*K88*K15*K24+2*K1*
K3*K88*K15*K23+2*K1*K3*K88*K15*K24+2*K1*K3*K89*K15*K23+2*K1*
K3*K89*K15*K24+2*K1*K3*K89*K15*K23+2*K1*K3*K89*K15*K24+2*K1*
K3*K89*K15*K23+2*K1*K3*K89*K15*K24+2*K1*K3*K90*K15*K23+2*K1*
K3*K90*K15*K24+2*K1*K3*K90*K15*K23+2*K1*K3*K90*K15*K24+2*K1*
K3*K90*K15*K23+2*K1*K3*K90*K15*K24+2*K1*K3*K91*K15*K23+2*K1*
K3*K91*K15*K24+2*K1*K3*K91*K15*K23+2*K1*K3*K91*K15*K24+2*K1*
K3*K91*K15*K23+2*K1*K3*K91*K15*K24+2*K1*K3*K92*K15*K23+2*K1*
K3*K92*K15*K24+2*K1*K3*K92*K15*K23+2*K1*K3*K92*K15*K24+2*K1*
K3*K92*K15*K23+2*K1*K3*K92*K15*K24+2*K1*K3*K93*K15*K23+2*K1*
K3*K93*K15*K24+2*K1*K3*K93*K15*K23+2*K1*K3*K93*K15*K24+2*K1*
K3*K93*K15*K23+2*K1*K3*K93*K15*K24+2*K1*K3*K94*K15*K23+2*K1*
K3*K94*K15*K24+2*K1*K3*K94*K15*K23+2*K1*K3*K94*K15*K24+2*K1*
K3*K94*K15*K23+2*K1*K3*K94*K15*K24+2*K1*K3*K95*K15*K23+2*K1*
K3*K95*K15*K24+2*K1*K3*K95*K15*K23+2*K1*K3*K95*K15*K24+2*K1*
K3*K95*K15*K23+2*K1*K3*K95*K15*K24+2*K1*K3*K96*K15*K23+2*K1*
K3*K96*K15*K24+2*K1*K3*K96*K15*K23+2*K1*K3*K96*K15*K24+2*K1*
K3*K96*K15*K23+2*K1*K3*K96*K15*K24+2*K1*K3*K97*K15*K23+2*K1*
K3*K97*K15*K24+2*K1*K3*K97*K15*K23+2*K1*K3*K97*K15*K24+2*K1*
K3*K97*K15*K23+2*K1*K3*K97*K15*K24+2*K1*K3*K98*K15*K23+2*K1*
K3*K98*K15*K24+2*K1*K3*K98*K15*K23+2*K1*K3*K98*K15*K24+2*K1*
K3*K98*K15*K23+2*K1*K3*K98*K15*K24+2*K1*K3*K99*K15*K23+2*K1*
K3*K99*K15*K24+2*K1*K3*K99*K15*K23+2*K1*K3*K99*K15*K24+2*K1*
K3*K99*K15*K23+2*K1*K3*K99*K15*K24+2*K1*K3*K100*K15*K23+2*K1*
K3*K100*K15*K24+2*K1*K3*K100*K15*K23+2*K1*K3*K100*K15*K24+2*K1*
K3*K100*K15*K23+2*K1*K3*K100*K15*K24+2*K1*K3*K101*K15*K23+2*K1*
K3*K101*K15*K24+2*K1*K3*K101*K15*K23+2*K1*K3*K101*K15*K24+2*K1*
K3*K101*K15*K23+2*K1*K3*K101*K15*K24+2*K1*K3*K102*K15*K23+2*K1*
K3*K102*K15*K24+2*K1*K3*K102*K15*K23+2*K1*K3*K102*K15*K24+2*K1*
K3*K102*K15*K23+2*K1*K3*K102*K15*K24+2*K1*K3*K103*K15*K23+2*K1*
K3*K103*K15*K24+2*K1*K3*K103*K15*K23+2*K1*K3*K103*K15*K24+2*K1*
K3*K103*K15*K23+2*K1*K3*K103*K15*K24+2*K1*K3*K104*K15*K23+2*K1*
K3*K104*K15*K24+2*K1*K3*K104*K15*K23+2*K1*K3*K104*K15*K24+2*K1*
K3*K104*K15*K23+2*K1*K3*K104*K15*K24+2*K1*K3*K105*K15*K23+2*K1*
K3*K105*K15*K24+2*K1*K3*K105*K15*K23+2*K1*K3*K105*K15*K24+2*K1*
K3*K105*K15*K23+2*K1*K3*K105*K15*K24+2*K1*K3*K106*K15*K23+2*K1*
K3*K106*K15*K24+2*K1*K3*K106*K15*K23+2*K1*K3*K106*K15*K24+2*K1*
K3*K106*K15*K23+2*K1*K3*K106*K15*K24+2*K1*K3*K107*K15*K23+2*K1*
K3*K107*K15*K24+2*K1*K3*K107*K15*K23+2*K1*K3*K107*K15*K24+2*K1*
K3*K107*K15*K23+2*K1*K3*K107*K15*K24+2*K1*K3*K108*K15*K23+2*K1*
K3*K108*K15*K24+2*K1*K3*K108*K15*K23+2*K1*K3*K108*K15*K24+2*K1*
K3*K108*K15*K23+2*K1*K3*K108*K15*K24+2*K1*K3*K109*K15*K23+2*K1*
K3*K109*K15*K24+2*K1*K3*K109*K15*K23+2*K1*K3*K109*K15*K24+2*K1*
K3*K109*K15*K23+2*K1*K3*K109*K15*K24+2*K1*K3*K110*K15*K23+2*K1*
K3*K110*K15*K24+2*K1*K3*K110*K15*K23+2*K1*K3*K110*K15*K24+2*K1*
K3*K110*K15*K23+2*K1*K3*K110*K15*K24+2*K1*K3*K111*K15*K23+2*K1*
K3*K111*K15*K24+2*K1*K3*K111*K15*K23+2*K1*K3*K111*K15*K24+2*K1*
K3*K111*K15*K23+2*K1*K3*K111*K15*K24+2*K1*K3*K112*K15*K23+2*K1*
K3*K112*K15*K24+2*K1*K3*K112*K15*K23+2*K1*K3*K112*K15*K24+2*K1*
K3*K112*K15*K23+2*K1*K3*K112*K15*K24+2*K1*K3*K113*K15*K23+2*K1*
K3*K113*K15*K24+2*K1*K3*K113*K15*K23+2*K1*K3*K113*K15*K24+2*K1*
K3*K113*K15*K23+2*K1*K3*K113*K15*K24+2*K1*K3*K114*K15*K23+2*K1*
K3*K114*K15*K24+2*K1*K3*K114*K15*K23+2*K1*K3*K114*K15*K24+2*K
```

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\$K1*K14*K18*K20-2*K2*K3*K17*K24-3*K2*K3*K10*K27+2*K2*K3*K20*K24-K2
\$K3*K20*K28-K2*K3*K21*K24-K2*K3*K22*K27-27-3*K2*K4*K17
\$K27-6*K2*K4*K18*K24+3*K2*K4*K13*K28+2*K2*K4*K20*K23-K2*K4*K21*
\$K23+3*K2*K4*K22*K24-K2*K4*K22*K28+2*K2*K4*K17*K24-K2*K4*K17*K28+2
\$K2*K6*K18*K23-K2*K6*K22*K23-K2*K6*K17*K24-K2*K6*K18*K23-K2*K6*K8*
\$K17*K23+K2*K6*K17*K27+3*K2*K8*K18*K24-K2*K8*K13*K27-K2*K8*K20*K23
\$K2*K8*K22*K24-4*K2*K9*K15*K23+2*K2*K9*K15*K27+6*K2*K9*K16*K24-2*
\$K2*K9*K16*K28+2*K2*K9*K17*K22-4*K2*K9*K18*K20+2*K2*K9*K18*K21
T221=2*
\$K2*K9*K20*K22+6*K2*K10*K15*K24-2*K2*K10*K15*K28+6*K2*K10*K16*K23-
\$K2*K10*K17*K20+2*K2*K10*K17*K21-6*K2*K10*K18*K22+2*K2*K10*K17*
\$2+6*K2*K10*K18*K22+2*K2*K10*K22*2+2*K2*K13*K15*K23+6*K2*K13*K17*K18
\$-2*K2*K13*K17*K22-2*K2*K14*K15*K24-2*K2*K14*K16*K23+2*K2*K14*K17*
\$K20+2*K2*K14*K18*K22-3*K2*K14*K18*K22+2*K2*K14*K15*K28+6*K2*K14*K16*
\$K27+2*K2*K15*K23-2*K2*K15*K27-4*K2*K15*K24+2*K2*K15*K16*
\$K28+2*K2*K16*K24+2*K2*K16*K15*K24+2*K2*K16*K16*K23-2*K2*K16*K16*
\$K27-3*K2*K15*K20-K3*K9*K16*K22-K3*K10*K15*K22-2*K2*K10*K16*K17+2
\$K23-K3*K10*K16*K20-K3*K10*K16*K21-K3*K10*K16*K20-K3*K10*K16*K18+K3
\$K13*K16*K22-K3*K14*K15*K18-K3*K14*K16*K20-4*K4*K6*K15*K24+2*K4*K
\$K6*K15*K28-4*K4*K6*K16*K23+2*K4*K7*K15*K24+2*K4*K7*K16*K23+2*K4*K
\$K8*K15*K23-2*K4*K8*K15*K27-6*K4*K8*K16*K24+2*K4*K8*K16*K28+2*K4*K
\$K9*K15*K28-2*K4*K9*K15*K22+2*K4*K9*K16*K20-4*K4*K9*K16*K21+2*K4*K10
\$K15*K20-K4*K10*K15*K21-6*K4*K10*K16*K18+3*K4*K10*K16*K22-3*K4*K
\$K13*K15*K18*K4*K13*K15*K22-3*K4*K13*K16*K17-K4*K14*K15*K17-K4*K14
\$K15*K20+3*K4*K14*K16*K18-K4*K14*K16*K22+2*K4*K14*K15*K24+2*K4*K16*K
\$K16*K23-K6*K9*K15*K17-K6*K9*K15*K20+2*K6*K9*K16*K18-K6*K9*K16*K22
+2*K6*K10*K15*K18-K6*K10*K15*K22+2*K6*K10*K16*K17+K6*K13*K15*K17
T321=-1*
\$K6*K14*K15*K18-K6*K14*K16*K17-K7*K9*K16*K17-K7*K10*K15*K18-K7*K10
\$K16*K17-2*K8*K9*K15*K18-K8*K9*K16*K17-K8*K9*K16*K20-K8*K10*K15*
\$K17-K8*K10*K15*K20+3*K8*K10*K16*K18-K8*K10*K16*K22+K8*K13*K15*K17
\$+K8*K13*K16*K17-K8*K14*K16*K18+4*K9*K10*K15*K22-2*K9*K10*K16*K22-4*
\$K9*K13*K15*K16+2*K9*K14*K16*K22-2*K9*K13*K15*K22+4*K9*K14*K15*K16
\$+4*K15*K16*K9*K22-6*K15*K16*K17+2*K15*K23*K16*K22+3*
\$K15*K27*K4*K2+2*K16*K24*K3*K2+6*K16*K24*K4*K2+K16*K24*K8*K2-3*
\$K16*K28*K4*K2+4*K23*K24*K1*2-6*K23*K24*K2+2*K23*K28*K2*2-2*
\$K24*K27*K1*2
T120 = 4*K1*K2*K23*K24-4*K1*K2*K24*K27-K1*K4*K18*K28-K1*K4*
\$K20*K23+K1*K4*K20*K27-K1*K4*K22*K24-K1*K6*K10*K23+K1*K6*K18*K27-
\$K1*K6*K20*K24-K1*K8*K18*K24-2*K1*K9*K16*K24+2*K1*K9*K18*K20-2*K1*
\$K10*K15*K24-2*K1*K10*K16*K23+2*K1*K10*K16*K27+2*K1*K10*K18*K22+K1
\$K17*K27-3*K1*K13*K16*K24-2*K1*K13*K18*K20+K1*K14*K16*K22-2*K2*K3
\$K18*K27+3*K2*K4*K20*K24-K2*K4*K20*K28-K2*K4*K21*K24-K2*K4*K22*K23
\$K28+4*K2*K6*K17*K22+2*K6*K17*K22-K2*K6*K18*K22-K2*K6*K18*K22+K6*
\$K20*K23-K2*K6*K22*K24-K2*K6*K18*K24-K2*K6*K18*K22+K6*K18*K22+2*
\$K2*K9*K18*K22+K2*K9*K20*K22-2*K2*K10*K15*K23+2*K2*K10*K15*K27+3*K2
\$K10*K16*K24-2*K2*K10*K16*K28+6*K2*K10*K17*K18-5*K2*K10*K18*K20+2
\$K2*K10*K18*K21+2*K2*K10*K20*K22+2*K2*K13*K15*K22+2*K2*K13*K16*
\$K23-2*K2*K13*K17*K20-2*K2*K13*K18*K22+K2*K13*K17*K22+4*K2*K15*K18
\$K22-2*K2*K14*K16*K24+2*K2*K14*K18*K22+6*K2*K14*K16*K24-2*K2*K16*K16
\$K27-3*K3*K10*K16*K18-K3*K13*K16*K17-K3*K13*K16*K20+2*K4*K6*K15*
\$K23-2*K4*K6*K15*K27-6*K4*K6*K16*K24+2*K4*K6*K16*K20+2*K4*K6*K17*K16
\$K24+2*K4*K8*K15*K24+2*K4*K8*K16*K23-2*K4*K8*K16*K27-K4*K9*K15*K20
\$-K4*K9*K16*K22-K4*K10*K15*K22-3*K4*K10*K16*K17+3*K4*K10*K16*K20
T220=-1*

\$K4*K10*K16*K21+K4*K13*K15*K20-4*K4*K13*K16*K18+K4*K13*K16*K22-K4*
\$K14*K15*K18-K4*K14*K16*K20+2*K6*K8*K16*K24-K6*K9*K15*K18-K6*K9*
\$K16*K20-K6*K10*K15*K20+3*K6*K10*K16*K18-K6*K10*K16*K22+K6*K13*K15
\$K18+K6*K15*K17-K6*K14*K16*K18-K7*K10*K16*K18-K7*K10*K16*K18+6*K9*K16*K18
\$K8*K10*K15*K18-K8*K10*K16*K20+K8*K15*K16*K18+4*K9*K10*K15*15-2*
\$K9*K13*K16*K22-4*K10*K13*K15*K16+2*K10*K14*K16*K22+K15*K24*K6*K22+
\$K15*K28*K4*K2+K16*K23+K6*K22+K16*K27*K3*2+4*K16*K27*K4*2-2*K23*
\$K27*K2*2+2*K24*K28*K2*2+K1*2*K24*K2*2+K2*2*K23*2-4*K2*2*K24*
\$2+K9*2*K16*2+K10*2*K15*2-4*K10*2*K16*2
T133 = 4*K1*K2*K23*K24-4*K1*K2*K23*K28+K1*K3*K16*K27+K1*K3*
\$K21*K24-K1*K3*K21*K28-K1*K3*K22*K27+K1*K4*K17*K27+K1*K4*K16*K24-
\$K1*K4*K18*K28+K1*K4*K21*K23-K1*K4*K22*K24+K1*K4*K22*K28+K1*K7*K17
\$K24-K1*K7*K17*K28+K1*K7*K18*K23-K1*K7*K22*K23-K1*K8*K17*K27-K1*
\$K8*K18*K24+K1*K8*K18*K23-K1*K8*K21*K23+K1*K8*K22*K24-K1*K8*K22*
\$K28-2*K1*K9*K15*K27-2*K1*K9*K16*K24+2*K1*K9*K16*K28-2*K1*K9*K18*
\$K21+2*K1*K9*K21*K22-2*K1*K10*K15*K24+2*K1*K10*K15*K28-2*K1*10*
\$K16*K23-2*K1*K10*K17*K21+2*K1*K10*K18*K22-K1*K10*K18*K22-K1*K10*
\$K22*2-2*K1*K13*K15*K23-2*K1*K13*K17*K18+2*K1*K13*K17*K22+2*K1*
\$K14*K15*K24-2*K1*K14*K15*K28+2*K1*K14*K16*K23+2*K1*K14*K17*K21-2*
\$K1*K14*K18*K22+K1*K14*K18*K22+2*K1*K14*K22+2*K1*K14*K27+K2*K3*
\$K18*K24-K2*K3*K18*K28+K2*K3*K21*K23-K2*K3*K22*K24+K2*K3*K22*K28+
\$K2*K4*K17*K24-K2*K4*K17*K28+K2*K4*K18*K23-K2*K4*K22*K23+K2*K4*K17
\$K23-K2*K8*K17*K24+K2*K8*K17*K28-K2*K8*K18*K23+K2*K8*K18*K22+2*K2
\$K9*K15*K24+2*K2*K9*K15*K28-2*K2*K9*K16*K23-2*K2*K9*K17*K21+2*K2*
\$K9*K16*K22-K2*K9*K18*K22-2*K2*K17*K22*2-2*K2*K17*K22+2*K2*K10*
\$K17*K18+2*K2*K10*K17*K22-K2*K13*K17*2+2*K2*K14*K15*K23
T233=2*K2*K14*
\$K17*K18-2*K2*K14*K17*K22-2*K3*K4*K10*K27-2*K3*K4*K16*K24+2*K3*K4*
\$K16*K28-2*K3*K7*K15*K24+2*K3*K7*K15*K28-2*K3*K7*K15*K23+3*K3*K7*
\$K15*K27+2*K3*K8*K16*K24-2*K3*K8*K16*K28+K3*K9*K16*K21+K3*K10*K15*
\$K21+K3*K10*K16*K18-K3*K10*K16*K22+K3*K13*K15*K18-K3*K13*K15*K22+
\$K3*K13*K16*K17-K3*K13*K15*K21-K3*K14*K16*K18+K3*K14*K16*K22-2*K4*
\$K7*K15*K23+2*K4*K9*K15*K24-2*K4*K8*K15*K28+2*K4*K8*K16*K22+K4*K9*
\$K15*K21+K4*K9*K16*K18-K4*K9*K16*K22+K4*K10*K15*K18-K4*K10*K15*K22
\$+K4*K10*K16*K17+K4*K13*K15*K17-K4*K14*K15*K18-K4*K14*K15*K22-K4*
\$K14*K16*K17+2*K7*K8*K15*K23+K7*K9*K15*K18-K7*K9*K15*K22+K7*K9*K16
\$K17+K7*K10*K15*K17-K7*K14*K15*K17-K7*K9*K15*K21-K7*K9*K16*K18+K7*
\$K9*K16*K22-K8*K10*K15*K18+K8*K10*K15*K22-K8*K10*K16*K17-K8*K13*
\$K15*K17+K8*K14*K15*K16-K8*K14*K15*K22+K8*K14*K16*K17+4*K9*K18*K15
\$K16+2*K9*K13*K15*K22-4*K9*K14*K15*K16-2*K10*K14*K15*2-K15*K24*K4
\$K22-K15*K24*K8*2+K15*K28*K4*2+K15*K28*K4*2-2*K16*K23*K4*2-K16*
\$K23*K8*2-2*K16*K27*K3*2+2*K23*K27*K1*2-2*K24*K2*2+K15*K24*
\$K22+K1*2*K28*2+K2*2*K23*2+K4*2*K16*2+K10*2*K15*2+K14*2*
\$K15*2
T132 = 8*K1*K2*K23*K27-8*K1*K2*K24*K28-4*K1*K2*K23*2+6*K1*
\$K2*K23*2+2*K1*K2*K28*2+K1*K3*K16*K28-K1*K3*K20*K27-K1*K3*K21*
\$K23+K1*K3*K21*K27+K1*K3*K22*K24-2*K1*K3*K22*K28+K1*K4*K17*K28-K1*
\$K4*K18*K23+3*K1*K4*K18*K27-K1*K4*K20*K27+K1*K4*K20*K28+2*K1*K4*
\$K21*K24-K1*K4*K21*K28+2*K1*K4*K22*K23-2*K1*K4*K22*K27-K1*K6*K17*
\$K27-K1*K6*K18*K22+K1*K6*K18*K28-K1*K6*K21*K23+K1*K6*K22*K24-K1*K6*
\$K22*K28-K1*K7*K17*K23+K1*K7*K17*K27+2*K1*K7*K18*K24-K1*K7*K18*K28-
\$K28-K1*K7*K20*K23-K1*K7*K22*K24+K1*K8*K17*K24-2*K1*K8*K17*K28+2*
\$K1*K8*K18*K23-2*K1*K8*K18*K27+K1*K8*K20*K24-K1*K8*K20*K28-K1*K8*
\$K21*K24-3*K1*K8*K22*K23+K1*K8*K22*K27+4*K1*K8*K22*K24-6*K1*K8*K25
\$K28+4*K1*K9*K15*K23-4*K1*K9*K16*K27+2*K1*K9*K17*K22-4*K1*K9*K18*
\$K22+2*K1*K9*K20*K21+K1*K9*K18*2+2*K1*K9*K22*2+2*K1*K10*K15*K23-
\$K1*K10*K15*K27-6*K1*K10*K16*K24+K1*K10*K16*K28-2*K1*K10*K17*
\$K16+2*K1*K10*K18*K20-4*K1*K10*K18*K21-2*K1*K10*K20*K22+2*K1*K10*

5K12K117K6K23+K1K1K6K17K2-7=2K1K1*6K4K1K1K2K24-2=K11K1K1K8K228-2K1*1Kd*
 5K20K23+K1K1K6K20K27-7=K11K4K2K1K24-6=K11K1K6K15K24-3=K4K1K1K9K115
 5K27+4K1K1K9K16K24-4=K4K1K9K16K2K28+2K11K1K9K17K22-2=K11K9K9K18
 5K20+2K1K1K9K5K18K22+4K1K1K9K20K22+4K1K1K10K15K24-4=K1K1K10K15
 5K28+2K1K1K1K10K16K23-6=K11K10K16K217+2K1K10K10K17K20-2=K1K10K17
 5K10K22+2K1K1K10K10K20K21-2=K1K10K1K18K22-4=K1K10K20K20*2+2K1K10K10
 5K22*2+4K1K1K113K15K23-2=K11K13K15K15K27-6=K1K13K16K24+2K1K1K1
 5K13K1K16K2K28-2=K1K1K13K17K22+2K1K1K13K16K20-2=K1K1K13K18K21K21
 T31=1-2*
 5K1K9K13K20K22-4K1K1K14K115K24-4=K1K1K14K16K2K23+2K1K1K14K16K27
 5+2K1K1K14K16K18K22+K11K15K20K22+2K1K2K3K17K217+3K22K3K18K24K24
 5=K2K3K2K3K20K27+K2K3K2K21K27-7K2K3K2K2K28+3K2K2K4K177K24+6=K2K2
 5K4K1K18K27-3=K2K4K4K20K24+2K2K4K4K20K28+3K2K4K4K21K24-4=K2K4K4
 5K21K28+2K2K4K4K22K23-3=K2K4K4K22K27-2=K2K4K4K17K27-3=K2K4K6K6
 5K18K24+2K2K6K6K18K28+K2K6K6K20K23-3=K2K6K6K21K23+2K2K6K6K28K22
 5K24-K2K6K6K22K28+K2K7K17K27+3K2K7K18K24-K2K7K18K27K18K28K22
 5K7K20K23K23-K2K7K1K22K24-K2K8K2K17K2K24+2K2K8K2K17K2K24-2=K2K8K2K17K2K24
 5K13K217+2K2K8K2K8K20K24-K2K8K2K20K27K23-K2K8K2K21K24-2=K2K8K2K22K22
 5K23+K2K8K2K22K22K27+4K2K9K9K15K24-4=K2K9K9K21K24-K2K9K2K9K9K16K2K23
 5-6K2K9K9K16K217-4=K2K9K9K9K18K22+2K2K9K9K20K27K21K24K9K2K20K27+2
 5K2K9K9K22K22+4K2K20K10K15K23-6=K2K9K9K10K15K2K27-1=2K2K10K16K2K24+
 5K6K2K10K10K16K2K24-6=K2K10K17K18+6K2K10K18K20-6=K2K10K18K20
 5K21-4K2K2K10K20K22+2K2K2K10K21K22-6=K2K13K1K15K24+2K2K2K13K1
 5K15K28-6=K2K2K13K16K23+4K2K2K13K17K20-2=K2K2K13K17K21+6=K2K2
 5K13K18K22-2=K2K2K13K17K22-6=K2K13K18K22-2=K2K13K18K22*2-4=K2K2
 5K16K15K23+2K2K2K14K15K24-7=6K2K2K14K16K16K24-2=K2K14K16K28+2K2
 5K2K14K17K22-4=K2K2K14K18K20+2K2K2K14K18K21+2K2K2K14K20K22+2K2K2
 5-6=K3K4K4K15K2K24+2K3K4K6K15K2K24+4K3K4K6K10K27-2=K3K4K7K16K16K22
 T31=1
 5K2K3K10K15K23-2=K3K4K6K15K27+2K3K4K6K16K2K24-2=K3K4K6K15K2K23+3K
 5K3K10K10K16K13K13K13K15K22+2K3K4K6K13K16K17-7=K3K3K4K6K13K16K20+
 5K3K4K6K13K16K21K-3K4K6K15K20K20K3K4K16K16K22-2=K4K6K6K15K2K3+4K4K6
 5K6K6K15K27+6=K4K6K6K6K16K24-4=K4K6K6K16K28+2=K4K6K4K17K15K23-3=K4K6
 5K7K15K2K27-6=K4K6K7K16K24+2K4K6K7K16K28-4=K4K6K7K15K2K24+2K4K6
 5K8K15K2K28-4=K4K6K8K10K23+6K4K9K9K10K27+K4K9K10K25K20-K4K9K9K9
 5K15K21+2K4K9K9K16K2K22+2K4K9K9K10K15K22+4=K4K9K10K16K217-3=K4K9
 5K10K16K20+3K4K9K9K10K16K212-2=K4K9K13K15K2K24+4=K13K15K21+6=K4K9
 5K13K16K1K17-3=K4K9K15K16K22+2K4K9K14K15K218-2=K4K9K14K14K15K2K22+2
 5K4K9K14K16K20-K4K9K14K16K2K21+2=K4K9K17K15K2K24+2=K4K9K17K16K28+2
 5K6K8K2K15K23-2=K4K9K6K8K15K27-4=K4K9K6K8K16K24+2=K4K9K6K16K2K24+K4
 5K9K15K18+2=K4K9K9K15K2K22+K4K9K13K16K20-K4K9K9K16K21+K4K9K10K15
 5K20K20-K4K9K10K15K211-3=K4K9K18K15K16K20+2K4K9K10K16K22-2=K4K9K13K
 5K15K2K16+K4K9K13K15K2K22-2=K4K9K13K16K20K17-K4K9K14K15K217-K4K9K14K15
 5K22+2=K4K9K14K16K21+K4K9K14K16K21+K4K9K15K2K22+2=K4K9K16K2K4K17K4K9K15
 5K18K2-K4K9K16K20-K4K9K16K20+3=K4K9K16K20K16K24-K4K9K16K24K17K4K9K22
 T41=1
 5K7K9K13K15K16K17K13K16K6K17-7K1K14K16K15K16-K3K5K9K15K17-2=K2K6
 5K9K9K15K20K22=K4K6K9K9K15K20-1=K20K22+2=K4K6K10K15K1K18-2=K4K6
 5K10K15K22+2K4K6K10K15K20K20-K3K5K10K16K21+K4K6K13K15K17K17+K3K13K13
 5K15K2K23-3=K4K6K13K16K2K13K3K5K13K16K22-2=K4K6K14K17K15K18-K4K6K14K
 5K16K217-K4K6K14K16K2K21-K4K6K15K15K16-K4K6K15K16K22+6=K4K9K13K13
 5K16K22+8=K4K9K14K15K16K16+12K10K15K15K16K16+4K10K16K14K15K22-6=K11
 5K14K16K22-2=K4K10K14K15K16K16-K15K22K4K6K22+2=K4K15K24K6K22-2=K4K15
 5K24K4K4K22+2K15K24K6K22-K16K23K4K6K22+2K16K23K4K6K22-2=K16K27K
 5K3K22-6=K16K27K4K22-K16K27K27K3K22-4=K16K27K27K1K22+6=K16K23K27K22
 5K22+4K24K24K23K1K22-6=K24K24K24K24K22+2K3K11K22K23K22-2=K24K24K24K24K22
 5K1K11K22K27K22-2=K24K24K24K23K27K22K22-2=K24K24K24K22+2K3K24K24K22
 5K15K22-2=K9K22K22K16K22-2=K10K22K22K15K24K6K10K22K22+2=K24K13K22K15

3K24-4*~~K4~~*K7*~~K16~~*K28+2*~~K4~~*~~K5~~*K15*~~K24~~-4*~~K4~~*K8*~~K15~~*~~K28~~+2*~~K4~~*~~K8~~*~~K16~~*

\$K23-6*K4*K8*K16*K27+K4*K9*K15*K21-K+*K9*K16*K22-K4*K10*K15*K22-K4
\$K10*K16*K17+K4*K10*K16*K20-J*K4*K10*K16*K21+K4*K13*K15*K20-2*K4*
\$K13*K15*K21-4*K4*K13*K15*K21+J*K4*K13*K16*K22-K4*K15*K21+K4*K15*K21+K4*
\$K14*K15*K22-K4*K14*K16*K20+2*K4*K14*K16*K21-2*K4*K15*K22+2*K4*
\$K7*K15*K28-2*K6*K7*K16*K23+2*K6*K15*K27+2*K6*K16*K24-2*K6*
\$K8*K16*K28+K6*K9*K16*K21+K6*K10*K15*K21+K6*K10*K16*K18-K6*K10*K15
\$K22+K6*K13*K15*K18-K6*K13*K15*K22+K6*K15*K16*K17-K6*K14*K15*K21-
\$K6*K14*K16*K18+K6*K16*K22+4*K7*K8*K15*K23-2*K7*K8*K15*K27-4*
\$K7*K8*K16*K24+2*K7*K8*K16*K28+K7*K9*K15*K18-2*K7*K9*K15*K22+K7*K9
\$K16*K20-K7*K9*K16*K21+K7*K10*K15*K20-K7*K10*K15*K21-3*K7*K10*K16
\$K18+2*K7*K10*K16*K22-2*K7*K13*K15*K18+K7*K13*K15*K22-2*K7*K13*
\$K16*K17-K7*K14*K15*K17-K7*K14*K15*K20+2*K7*K14*K16*K18-K7*K14*K16
\$K22-2*K8*K9*K15*K21-K8*K9*K16*K18+2*K8*K9*K16*K22-K8*K10*K15*K18
+2*K8*K10*K15*K22-K8*K13*K16*K20+2*K8*K10*K16*K21-K8*K13*K16*K17-
\$K8*K13*K15*K20+K8*K13*K15*K21+3*K8*K13*K16*K18-2*K8*K13*K16*K22+2
\$K8*K14*K15*K18-3*K8*K14*K15*K22+K8*K14*K16*K17+K8*K14*K16*K20-K8
\$K14*K16*K21+4*K9*K10*K15*K16+4*K9*K13*K15*K22-6*K9*K13*K16*K22
T442=-8*
\$K9*K14*K15*K16-12*K10*K15*K15*K16-4*K10*K14*K15*K22+6*K10*K14*K16
\$K22+2*K10*K14*K15*K16+K15*K24-K7*2-2*K15*K24*K22+K15*K28*K4*2
\$+3*K15*K28*K8*2+K16*K23*K7*2-2*K16*K23*K8*2+K16*K27*K3*2+4*
\$K16*K27*K4*2+2*K16*K27*K8*2+4*K23*K27*K1*2-6*K23*K27*K2*2-4*
\$K26*K28*21*2+6*K24*K28*K2*2+K1*2*2*K24*2*2-2*K1*2*2*K27*2*2+3*K1*2*2
\$K28*2*2+K2*2*2*K23*2*2-4*K2*2*2*K24*2*2-2*K2*2*2*K28*2*2+K9*K27*K16*2*2+
\$K10*2*2*K15*2*2-4*K10*2*2*K16*2*2-2*K16*2*2+3*K16*2*2*K15*2*2+
\$K14*2*2*K16*2*2
T141 = 8*K1*K2*K23*K27-8*K1*K2*K24*K26-2*K1*K2*K23*2*2-6*K1*
\$K2*K27*2*2+4*K1*K2*K28*2*2-K1*K3*K22*K28+K1*K4*K20*K28-K1*K4*K21*
\$K28+K1*K4*K22*K23-2*K1*K4*K22*K27+K1*K6*K13*K28-K1*K6*K20*K27-K1*
\$K6*K21*K6*K23+K1*K6*K21*K27+K1*K6*K22*K28-K28-K1*K6*K21*K8*
\$K28-K1*K6*K20*K23+K1*K7*K20*K27-K1*K7*K22*K24-K1*K8*K17*K28+K1*K8
\$K18*K23-2*K1*K8*K18*K27+K1*K8*K20*K24-2*K1*K8*K20*K28-K1*K8*K21*
\$K24-3*K1*K8*K22*K23+2*K1*K8*K22*K27+2*K1*K9*K15*K24-6*K1*K9*K15*
\$K26+2*K1*K9*K16*K23-4*K1*K9*K16*K27-2*K1*K9*K18*K22+2*K1*K9*K20*
\$K21+3*K1*K9*K22*2+2*K1*K10*K15*K23-4*K1*K10*K15*K27+4*K1*K10*K16
\$K28-2*K1*K10*K20*K22+2*K1*K10*K21*K22-4*K1*K13*K15*K24+4*K1*K13*
\$K15*K28-4*K1*K13*K16*K23+6*K1*K13*K16*K27+4*K1*K13*K18*K22-2*K1*
\$K13*K20*K21+K1*K13*K20*2-2*K1*K13*K22*2-6*K1*K14*K15*K23+4*K1*
\$K14*K15*K27+4*K1*K14*K16*K24-4*K1*K14*K16*K28+2*K1*K14*K17*K22-2*
\$K1*K14*K18*K20+2*K1*K14*K18*K21+4*K1*K14*K20*K22+2*K1*K14*K22*K26+
\$K2*K16*K20*K26-K2*K6*K21*K28+K2*K6*K22*K23-2*K2*K6*K22*K27-K2*K7*
\$K20*K28-K2*K7*K22*K23+K2*K7*K22*K28+K2*K8*K28+K2*K8*K20*K23
\$-2*K2*K8*K20*K27-K2*K8*K21*K27+2*K2*K8*K22*K28+K2*K8*K24-2*K2*
\$K8*K22*K28+2*K2*K9*K15*K23-4*K2*K9*K15*K27+4*K2*K9*K16*K28-2*K2*
\$K9*K20*K22+2*K2*K9*K21*K22+4*K2*K10*K15*K28-2*K2*K10*K22*2
T241=-4*K2*
\$K13*K15*K23+6*K2*K13*K15*K27-6*K2*K13*K16*K28+4*K2*K13*K20*K22-2*
\$K2*K13*K21*K22+4*K2*K14*K15*K24-4*K2*K14*K15*K28+4*K2*K14*K16*K23
\$-6*K2*K14*K16*K27-4*K2*K14*K18*K22+2*K2*K14*K20*K21-K2*K14*K20*K22+
\$+2*K2*K14*K22*2+2*K3*K3*K15*K23-3*K3*K15*K22-2*K4*K6*K15*K28+2*
\$K4*K7*K15*K28-2*K4*K6*K15*K23+4*K4*K6*K15*K27-4*K4*K6*K16*K28+K4
\$K9*K15*K22-2*K4*K13*K15*K22+K4*K14*K15*K20-K4*K14*K15*K21+2*K4*
\$K14*K16*K22+2*K6*K7*K15*K23-2*K6*K7*K15*K27+2*K6*K7*K16*K28-2*K6*
\$K8*K15*K24+4*K6*K8*K15*K26-2*K6*K8*K16*K23+4*K6*K8*K16*K27-K6*K9*
\$K15*K21+K6*K9*K16*K22+K6*K10*K15*K22-K6*K13*K15*K20+K6*K13*K15*
\$K21-2*K6*K13*K16*K22+K6*K14*K15*K18-2*K6*K14*K15*K22+K6*K14*K16*
\$K20-K6*K14*K16*K21+2*K7*K8*K15*K24+2*K7*K8*K16*K23-2*K7*K8*K16*
\$K27-K7*K9*K15*K27-K7*K9*K16*K22-K7*K10*K15*K22+7*K13*K15*K20+K7*

\$K13*K16*K22-K7*K14*K15*K18-K7*K14*K16*K20+4*K9*K15*K18-3*K9*K9*
\$K15*K22+K8*K9*K16*K20-K9*K9*K15*K21+K8*K10*K15*K20-K8*K10*K15*K21
\$+2*K8*K10*K16*K22-2*K8*K18+2*K8*K13*K15*K22-2*K8*K13*K16*
\$K20+K8*K13*K16*K21-K8*K14*K15*K20+2*K8*K14*K16*
\$K18-2*K8*K14*K16*K22-2*K9*K10*K15*K22+2*K9*K13*K15*K16+4*K9*K14*
\$K15*2-4*K9*K14*K16*2+4*K10*K13*K15*2-8*K10*K14*K15*K16+4*K13*
\$K14*K15*2+6*K13*K14*K16*2-2*K15*K16*K9*2-6*K15*K16*K13*2
T341=4*
\$K15*K16*K14*2+3*K15*K23*K6*2+K15*K27*K6*2-2*K15*K27*K6*2-2*
\$K16*K24*K8*2-K16*K26*K6*2+2*K16*K28*K8*2-2*K23*K24*K1*2+6*K23
\$K28*K1*2-4*K23*K28*K2*2+4*K24*K27*K1*2-4*K27*K28*K1*2+6*K27*
\$K28*K2*2
T40 = 4*K1*K2*K23*K23-4*K1*K2*K27*K28-K1*K4*K22*K20-K1*K6*
\$K20*K26-K1*K6*K22*K23+K1*K6*K22*K27-K1*K6*K18*K28-K1*K6*K20*K23+
\$K1*K8*K20*K27-K1*K8*K22*K24-2*K1*K9*K15*K23+2*K1*K9*K15*K27-2*K1*
\$K9*K16*K28+2*K1*K9*K20*K22-2*K1*K10*K15*K28+K1*K10*K22*2+2*K1*
\$K13*K15*K23-2*K1*K13*K15*K27+2*K1*K13*K16*K28-2*K1*K13*K20*K22-2*
\$K1*K14*K15*K24-2*K1*K14*K16*K23+2*K1*K14*K16*K27+2*K1*K14*K18*K22
\$+K1*K14*K20*2-K2*K6*K22*K28-K2*K8*K20*K28-K2*K8*K22*K23+K2*K8*
\$K22*K27-2*K2*K9*K15*K24+2*K9*K22*2+2*K2*K13*K15*K28-K2*K13*K22
\$K22-2*K2*K14*K15*K25+2*K2*K14*K15*K27-2*K2*K14*K16*K28+2*K2*K14*
\$K20*K22+2*K4*K8*K15*K28-K4*K14*K15*K22+2*K6*K8*K15*K23-2*K6*K8*
\$K15*K27+2*K6*K8*K16*K28-K6*K9*K15*K22+K6*K13*K15*K22-K6*K14*K15*
\$K20-K6*K14*K16*K22-K6*K9*K15*K20-K8*K9*K16*K22-K8*K10*K15*K22+K8*
\$K13*K15*K20+K8*K13*K16*K22-K8*K14*K15*K18-K8*K14*K16*K20-2*K9*K13
\$K15*2+4*K9*K14*K15*K16+2*K10*K14*K15*2-4*K13*K14*K15*K16*K15*
\$K24*K8*2+K1*K5*K28*K6*2+K16*K23*K8*2-K16*K27*K8*2-2*K23*K27*K1
\$K22+2*K24*K28*K1*2+K1*2*2*K23*2*2+K1*2*2*K27*2*2+K2*2*2*K28*2*2+K9*2*2
\$K15*2+K13*2*2*K15*2+K14*2*2*K10*2*2
T153 = 4*K1*K2*K24*K27-4*K1*K2*K27*K20+K1*K4*K21*K27+K1*K7*
\$K18*K27+K1*K7*K21*K24-K1*K7*K21*K28-K1*K7*K22*K27-K1*K8*K21*K27-2*
\$K1*K10*K16*K27-K1*K10*K21*2-2*K1*K13*K15*K27-2*K1*K13*K16*K22+2*
\$K1*K13*K16*K28-2*K1*K13*K18*K21+2*K1*K13*K21*K22+2*K1*K14*K16*
\$K27+K1*K14*K21*2+2*K2*K3*K21*K27+K2*K4*K18*K27+K2*K4*K21*K24-K2*K4
\$K21*K28-K2*K4*K22*K27+K2*K7*K17*K27+K2*K7*K18*K24-K2*K7*K18*K28+
\$K2*K7*K21*K23-K2*K7*K22*K24+K2*K7*K22*K28-K2*K8*K18*K27-K2*K8*K21
\$K24+K2*K8*K21*K28+K2*K8*K22*K27-2*K2*K9*K16*K27-K2*K9*K21*2-2*
\$K2*K10*K15*K27-2*K2*K10*K16*K24+2*K2*K10*K16*K28-2*K2*K10*K16*K21
\$+2*K2*K10*K21*K22-2*K2*K13*K15*K24+2*K2*K13*K15*K28-2*K2*K13*K16*
\$K23-2*K2*K13*K17*K21+2*K2*K13*K18*K22-K2*K13*K18*2-K2*K13*K22*2
\$+2*K2*K14*K15*K27+2*K2*K14*K16*K24-2*K2*K14*K16*K28+2*K2*K14*K18*
\$K21-2*K2*K14*K21*2-2*K3*K7*K16*K27+K3*K13*K15*K21-2*K4*K7*K15*
\$K27-2*K4*K7*K16*K24+2*K4*K7*K16*K28+2*K4*K8*K16*K27+K4*K10*K16*
\$K21+K4*K13*K15*K21+K4*K13*K16*K18-K4*K13*K16*K22-K4*K14*K16*K21+2
\$K7*K8*K15*K27+2*K7*K8*K16*K24-2*K7*K8*K16*K27+K7*K8*K16*K21+K7*
\$K10*K15*K21+K7*K10*K16*K18-K7*K10*K16*K22+K7*K13*K15*K18-K7*K13*
\$K15*K22*K7*K13*K16*K17-K7*K14*K15*K21-K7*K14*K16*K18+K7*K14*K16*
\$K22-K8*K10*K16*K21-K8*K13*K15*K21-K8*K13*K16*K18+K8*K13*K16*K22
T253 =
\$K8*K14*K16*K21+2*K9*K13*K16*2+4*K10*K13*K15*K16-2*K10*K14*K16*2
\$-4*K13*K14*K15*K16-K15*K24*K7*2+K15*K28*K7*2-K16*K23*K7*2-K15*
\$K27*K4*2-K16*K27*K8*2+2*K23*K27*K2*2-2*K24*K28*K2*2+2*K27*
\$K2*2+2*K24*K2*2+2*K2*2*2*K26*2+K10*2*2*K15*2+K13*2*2*K15*2+K14*2*2
\$K16*2
T152 = -4*K1*K2*K23*K27+4*K1*K2*K24*K26+5*K1*K2*K27*2-4*K1
\$K2*K20*2+2*K1*K4*K21*K4*2+K1*K4*K22*K27-K1*K6*K21*K27+K1*K7*K16*
\$K28-K1*K7*K20*K27-K1*K7*K21*K23+K1*K7*K21*K27+K1*K7*K22*2-K1*
\$K7*K22*K23+K1*K8*K10*K27+K1*K8*K21-K8*K21-K8*K21*K28-2*K1*K8*

\$K22*K27+2*K1*K9*K16*K27+K1*K9*K21*2+2*K1*K10*K15*K27-2*K1*K10*
\$K16*K28-2*K1*K15*K21*K22+2*K1*K15*K24-4*K1*K13*K15*K28+2*K1*
\$K13*K16*K23-6*K1*K13*K15*K27-2*K1*K13*K16*K22+2*K1*K13*K20*K21-K1
\$K13*K21*2+2*K1*K13*K22*2-4*K1*K14*K15*K27-2*K1*K14*K16*K24+4*
\$K1*K14*K16*K28-2*K1*K14*K16*K21+4*K1*K14*K21*K22+2*K1*K14*K27*K2
\$K3*K21*K24+K2*K4*K17*K27+K2*K4*K18*K24-K2*K4*K20*K27+3*K2*K4*K21
\$K27-K2*K4*K22*K28-K2*K4*K18*K27-K2*K6*K21*K24+K2*K6*K21*K28+K2*
\$K6*K22*K27+K2*K7*K17*K24+3*K2*K7*K18*K27-K2*K7*K20*K24+K2*K7*K20*
\$K28+2*K2*K7*K21*K24-K2*K7*K21*K28+K2*K7*K22*K23-2*K2*K7*K22*K27-
\$K2*K8*K18*K28+K2*K8*K20*K27-K2*K8*K21*K23-2*K2*K8*K21*K27-K2*K8*
\$K22*K24+2*K2*K8*K22*K28+2*K2*K9*K15*K27-2*K2*K9*K16*K28-2*K2*K9*
\$K21*K22-2*K2*K10*K15*K28-8*K2*K10*K16*K27-2*K2*K10*K17*K21+2*K2*
\$K10*K20*K21-K2*K10*K18*2-2*K2*K10*K21*2+K2*K10*K22*2+2*K2*K13*
\$K15*K23-6*K2*K13*K15*K27-8*K2*K13*K16*K24+6*K2*K13*K16*K28-2*K2*
\$K13*K17*K18+2*K2*K13*K16*K20-6*K2*K13*K16*K21-2*K2*K13*K20*K22
T252=4*
\$K2*K13*K21*K22-2*K2*K14*K15*K24+4*K2*K14*K15*K28-2*K2*K14*K16*K23
+6*K2*K14*K16*K27+2*K2*K14*K18*K22-2*K2*K14*K20*K21+K2*K14*K21*2
-2*K2*K14*K22*2-2*K3*K7*K16*K27-2*K3*K7*K16*K24+K3*K10*K16*K21+
\$K3*K13*K16*K13+2*K4*K6*K16*K27-2*K4*K7*K15*K28-6*K4*K7*K16*K27-2*
\$K4*K8*K15*K27+2*K4*K8*K16*K28+K4*K10*K16*K18*K4*K13*K15*K22+K4*
\$K13*K16*K17-K4*K13*K16*K20+3*K4*K13*K16*K21+K4*K14*K15*K21-K4*K14
\$K16*K22+2*K6*K7*K15*K27+2*K6*K7*K16*K24-2*K6*K7*K16*K27-K4*K6*K8*
\$K16*K27-K6*K10*K16*K21-K6*K13*K15*K21-K6*K13*K16*K18+K6*K13*K16*
\$K22+K6*K14*K16*K21-2*K7*K8*K15*K24+4*K7*K8*K15*K28-2*K7*K8*K16*
\$K23+4*K7*K8*K16*K27-K7*K9*K15*K21+K7*K9*K16*K22+K7*K10*K15*K22+K7*
\$K10*K16*K17-K7*K10*K16*K20+K7*K10*K16*K21-K7*K13*K15*K20+K7*
\$K13*K15*K21+3*K7*K13*K16*K18-2*K7*K13*K16*K22+K7*K14*K15*K18-2*K7*
\$K14*K15*K22+K7*K14*K16*K20-K7*K14*K16*K21+K8*K9*K16*K21+K8*K10*
\$K15*K21-K8*K10*K16*K22+K8*K13*K15*K18-2*K8*K13*K15*K22+K8*K13*K16
\$K20-2*K8*K13*K16*K21-2*K8*K14*K15*K21-K8*K14*K16*K18+K8*K14*
\$K16*K22-4*K9*K13*K15*K16+2*K9*K14*K16*K22-2*K9*K13*K15*K22+8*K10*
\$K13*K16*2+4*K10*K14*K15*K16+4*K13*K14*K15*2-6*K13*K14*K16*2+6*
\$K15*K16*K13*K22-4*K15*K16*K14*2+K15*K23*K7*2-K15*K27*K7*2
T352=2*K15
\$K27*K8*2-K16*K24*K4*2-2*K16*K24*K7*2+K16*K24*K8*2+K16*K28*K7
\$*2-2*K16*K28*K8*2+2*K23*K28*K2*2-2*K24*K27*K1*2+8*K24*K27*K2
\$*2+4*K27*K28*K1*2-6*K27*K28*K2*2
T151 = -4*K1*K24*K23*K28+8*K1*K22*K27*K28+K1*K4*K22*K28-K1*K6
\$K21*K28-K1*K6*K22*K27-K1*K7*K28-K1*K7*K22*2+K1*K7*K22*K27+
\$K1*K8*K16*K28-K1*K8*K20*K27-K1*K8*K21*K23+K1*K9*K21*K27+K1*K8*K22
\$K24-3*K1*K8*K22*K28-2*K1*K9*K15*K27+2*K1*K9*K16*K28+2*K1*K9*K21*
\$K22+2*K1*K10*K15*K28-K1*K10*K22*2-2*K1*K13*K15*K23+4*K1*K13*K15*
\$K27-4*K1*K13*K16*K28+2*K1*K13*K20*K22-2*K1*K13*K21*K22+2*K1*K14*
\$K15*K24-6*K1*K14*K15*K28+2*K1*K14*K16*K23-4*K1*K14*K16*K27-2*K1*
\$K14*K18*K22+2*K1*K14*K20*K21+3*K1*K14*K22*2+K2*K3*K17*K27+2*K3*
\$K18*K24-K2*K3*K20*K27+2*K2*K3*K21*K27+K2*K4*K17*K24+4*K2*K4*K18*
\$K27-K2*K4*K20*K24+3*K2*K4*K21*K24-K2*K6*K17*K27-K2*K6*K18*2+K2*
\$K6*K20*K27-2*K2*K6*K21*K27+K2*K6*K22*K28+2*K2*K7*K17*K27+3*K2*K7*
\$K18*K24-2*K2*K7*K20*K27+K2*K7*K21*K27-K2*K7*K22*K28+K2*K8*K20*K28
-K2*K8*K21*K28+K2*K8*K22*K23-2*K2*K8*K22*K27+2*K2*K9*K15*K28-K2*
\$K9*K22*2-8*K2*K10*K16*K24-2*K2*K10*K17*K18+2*K2*K10*K18*K20-6*K2
\$K10*K18*K21-4*K2*K13*K15*K28-12*K2*K13*K16*K27+2*K2*K13*K17*K20-
\$K2*K13*K17*K21+4*K2*K13*K20*K21-K2*K13*K17*2-4*K2*K13*K18*2-
\$K2*K13*K20*2-K2*K13*K21*2+2*K2*K13*K22*2+2*K2*K14*K15*K23-4*K2
\$K14*K15*K27+4*K2*K14*K16*K28-2*K2*K14*K17*K22+2*K2*K14*K21*K22-2*
\$K3*K4*K16*K24+2*K3*K4*K16*K27-4*K3*K4*K16*K27+K3*K10*K16*K18

T251=K3*

\$K13*K16*K17-K3*K13*K16*K20+2*K3*K13*K16*K21+2*K4*K6*K16*K24-8*K4*
\$K7*K16*K24-2*K4*K8*K15*K28+K4*K10*K16*K17-K4*K10*K16*K20+3*K4*K10
\$K16*K21+4*K4*K13*K16*K18+K4*K14*K15*K22+2*K6*K7*K15*K28+4*K6*K7*
\$K16*K27+2*K6*K8*K15*K27-2*K6*K8*K16*K28-K6*K10*K16*K18-K6*K13*K15
\$K22-K6*K13*K16*K17+K6*K13*K16*K20-2*K6*K13*K16*K21-K6*K14*K15*
\$K21+K6*K14*K16*K22+2*K7*K8*K15*K23-2*K7*K8*K15*K27+2*K7*K8*K16*
\$K28-K7*K9*K15*K22+2*K7*K10*K16*K18+K7*K13*K15*K22+2*K7*K13*K16*
\$K17-2*K7*K13*K16*K20+K7*K13*K16*K21-K7*K14*K15*K20-K7*K14*K16*K22
-K8*K9*K15*K21+K8*K9*K16*K22+K8*K10*K15*K22-K8*K13*K15*K20+K8*K13
\$K15*K21-2*K8*K13*K16*K22+K8*K14*K15*K18-3*K8*K14*K15*K22+K8*K14*
\$K16*K20-K8*K14*K16*K21+2*K9*K13*K15*2-4*K9*K14*K15*K16-2*K10*K14
\$K15*2+8*K13*K14*K15*K16-K15*K24*K8*2+3*K15*K28*K8*2-K16*K23*
\$K4*2-K16*K27*K8*2-4*K16*K27*K4*2-K16*K27*K6*2-K16*K27*K7*2+2*
\$K16*K27*K8*2+2*K23*K27*K1*2-2*K24*K28*K1*2-2*K1*2*2*K27*2+3*
\$K1*2*2*K8*2+4*K2*2*K24*2+6*K2*2*K27*2-2*K2*2*K28*2+4*K10*2
\$2*K16*2-2*K13*2*K15*2+6*K13*2*K16*2+3*K14*2*K15*2-2*K14*2*2
\$K16*2*2
T50 = 2*K1*K2*K28*2-K1*K6*K22*K28-K1*K8*K20*K28-K1*K8*K22

\$K23+K1*K8*K22*K27-2*K1*K9*K15*K23+K1*K9*K22*2-2*K1*K13*K15*K28-
\$K1*K13*K22*2-2*K1*K14*K15*K23+2*K1*K14*K15*K27-2*K1*K14*K16*K28+
\$2*K1*K14*K16*K22-2*K16*K22-K20-2*K2*K14*K15*K28+K2*K15*K22*2+2*
\$K6*K8*K15*K28-K5*K14*K15*K22-K8*K9*K15*K22+K8*K13*K15*K22-K8*K14*
\$K15*K20-K8*K14*K16*K22+2*K9*K14*K15*2-2*K13*K14*K15*2+2*K15*K16*
\$K14*2+K15*K23*K8*2-K15*K27*K8*2+K16*K28*K8*2+2*K23*K28*K1*2
-2*K27*K28*K1*2
T63 = -2*K1*K2*K27*2-K1*K7*K21*K27+2*K1*K13*K16*K27+K1*
\$K13*K21*2+K2*K7*K21*K28+K2*K7*K22*K27+K2*K8*K21*K27+2*K2*K13*K15
\$K27-2*K2*K13*K15*K28-2*K2*K13*K21*K22-2*K2*K14*K16*K27-2*K2*14*
\$K21*2-2*K7*K8*K16*K27-K7*K13*K15*K21+K7*K13*K16*K22+K7*K14*K16*
\$K21+K8*K13*K16*K21+2*K13*K14*K16*2-2*K15*K16*K13*2+2*K15*K17*K7*
\$2-K16*K28*K2*2+2*K27*K28*K2*2
T62 = -4*K1*K2*K27*K28-K1*K7*K21*K28-K1*K7*K22*K27-K1*K8*
\$K21*K27-2*K1*K13*K15*K27+2*K1*K13*K16*K26+2*K1*K13*K21*K22+2*K1*
\$K14*K16*K27*K15*K16*K21*2+2*K2*K7*K22*K28+K2*K8*K21*K28+K2*K8*K22*
\$K27+2*K2*K13*K15*K28-K2*K13*K22*2+2*K2*K14*K15*K27-2*K2*K14*K16*
\$K28-2*K2*K14*K21*K22+2*K7*K8*K15*K27-2*K7*K8*K16*K28-K7*K13*K15*
\$K22-K7*K14*K15*K21+K7*K14*K16*K22-K8*K13*K15*K21+K8*K13*K16*K22+
\$K8*K14*K16*K21-4*K13*K14*K15*K16+K15*K28*K7*2-K16*K27*K8*2+K1*2*
\$2*K27*2+K2*2*K28*2+K13*2*K15*2+K14*2*K16*2
T61 = -2*K1*K2*K28*2-K1*K7*K22*K28-K1*K8*K21*K28-K1*K8*
\$K22*K27-2*K1*K13*K15*K23+K1*K13*K22*2-2*K1*K14*K15*K27+2*K1*K14*
\$K16*K28+2*K1*K14*K21*K22+K2*K8*K22*K28+2*K2*K14*K15*K28-K2*K14*
\$K22*2+2*K7*K8*K15*K28-K7*K14*K15*K22-K8*K13*K15*K22-K8*K14*K15*
\$K21+K8*K14*K16*K22+2*K14*K15*2-2*K15*K16*K14*2+2*K15*K27*K8*2
\$2-K16*K28*K2*2+2*K27*K28*K1*2
T60 = -K1*K8*K22*K28-2*K1*K14*K15*K26+K1*K14*K22*2-K8*K14

\$K15*K22+K15*K28*K8*2+K1*2*2*K23*2+K14*2*K15*2
T11=T111+T211
T22=T122+T222+T322
T21=T121+T221+T321
T20=T120+T220
T33=T133+T233
T32=T132+T232+T332+T432
T31=T131+T231+T331+T431
T30=T130+T230
T43=T143+T243
T42=T142+T242+T342+T442
T41=T141+T241+T341

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T53=T153+T253
T52=T152+T252+T352
T51=T151+T251
TT(1)=T03
TT(2)=T02
TT(3)=T01
TT(4)=T00
TT(5)=T13
TT(6)=T12
TT(7)=T11
TT(8)=T10
TT(9)=T23
TT(10)=T22
TT(11)=T21
TT(12)=T20
TT(13)=T33
TT(14)=T32
TT(15)=T31
TT(16)=T30
TT(17)=T43
TT(18)=T42
TT(19)=T41
TT(20)=T40
TT(21)=T53
TT(22)=T52
TT(23)=T51
TT(24)=T50
TT(25)=T63
TT(26)=T62
TT(27)=T61
TT(28)=T60
499  FORMAT(/,2X,'FOR THE SIXTH DEGREE POLYNOMIAL,')
500  FORMAT(2X,'T',2I1,' : ',F20.8)
      WRITE(6,499)
      IG=1
      DO 502 III=1,7
      IF=III-1
      DO 501 II=1,4
      IS=4-II
      WRITE(6,500) IF,IS,TT(IG)
      IG=IG+1
501  CONTINUE
502  CONTINUE
      RETURN
      END

DIR. COS. OF SCREW $12 ARE: ( 0.3600, 0.0965, 0.9279 )
ROT. ANG. OF SCREW $12 IS:133.200

DIR. COS. OF SCREW $13 ARE: ( 0.1457,-0.0390, 0.9886 )
ROT. ANG. OF SCREW $13 IS: 70.600

DIR. COS. OF SCREW $14 ARE: ( 0.4267,-0.2464, 0.8702 )
ROT. ANG. OF SCREW $14 IS: 87.900

DIR. COS. OF SCREW $15 ARE: ( 0.4027,-0.0251,-0.9150 )
ROT. ANG. OF SCREW $15 IS: 25.200

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THE COORDINATES OF A POINT ON SCREW \$12 ARE: (0.9414, 0.5214,-0.4194)

THE COORDINATES OF A POINT ON SCREW \$13 ARE: (-0.6103, 0.8371, 0.1230)

TRANSFORMED COORDINATE SYSTEM

X-AXIS: (0.7818, 0.5116,-0.3565)

Y-AXIS: (-0.5051, 0.8538, 0.1087)

Z-AXIS: (0.3600, 0.0965, 0.9279)

DIR. COS. OF \$12 IN THE TRANSFORMED COORDINATE SYSTEM: (0.0 , 0.0 , 1.0000)

DIR. COS. OF \$13 IN THE TRANSFORMED COORDINATE SYSTEM: (-0.2585, 0.0 , 0.9660)

DIR. COS. OF \$14 IN THE TRANSFORMED COORDINATE SYSTEM: (-0.1027,-0.3330, 0.9373)

DIR. COS. OF \$15 IN THE TRANSFORMED COORDINATE SYSTEM: (0.6282,-0.3259,-0.7065)

ROTHS B. LINES - DIR. COS. = 0.1489 -0.0502 0.9876

TRANS B. LINES - DIR. COS.= -0.2614 -0.0113 0.9652

CUBIC1 = -0.0000 CUBIC2 = -0.0000

SYLVESTERS DYALITIC ELIMINANT = -0.0000

ROTHS B. LINES - DIR. COS. = 0.3658 0.0922 0.9261

TRANS B. LINES - DIR. COS.= 0.0030 -0.0068 1.0000

CUBIC1 = -0.0000 CUBIC2 = -0.0000

SYLVESTERS DYALITIC ELIMINANT = 0.0000

ROTHS B. LINES - DIR. COS. = 0.4713 0.7098 0.5230

TRANS B. LINES - DIR. COS.= 0.5455 0.4227 0.7237

CUBIC1 = -0.0000 CUBIC2 = 0.0001

SYLVESTERS DYALITIC ELIMINANT = -0.0000

ROTHS B. LINES - DIR. COS. = 0.5040 -0.4612 0.7302

TRANS B. LINES - DIR. COS.= -0.1022 -0.5710 0.8145

CUBIC1 = 0.0000 CUBIC2 = -0.0001

SYLVESTERS DYALITIC ELIMINANT = 0.0000

FOR THE SIXTH DEGREE POLYNOMIAL,

T03 : 0.01771640
T02 : -0.07694066
T01 : 0.03894479
T00 : 0.00069168
T13 : 0.20106225
T12 : 0.06664660
T11 : -0.18264199
T10 : 0.04021714
T23 : -0.73521586
T22 : 0.21562524
T21 : 0.09921132
T20 : -0.10442553
T33 : 0.33546856
T32 : -0.26335464
T31 : 0.69709206
T30 : -0.15320699
T43 : -0.46760378
T42 : 0.69115853
T41 : -0.22859391
T40 : 0.15475631
T53 : 0.13377606
T52 : -0.77759246
T51 : 0.61789977
T50 : -0.04946211
T63 : 0.00102496
T62 : 0.13297683
T61 : -0.30818613
T60 : 0.26050377

ROOTS OF THE SIXTH DEGREE POLYNOMIAL

REAL ROOTS	IMAGINARY ROOTS
20.52583	0.0
3.16282	0.0
0.33094	0.71135
0.33094	-0.71135
-0.36402	0.0
0.18347	0.0

THE COEFFICIENTS OF THE SIXTH DEGREE POLYNOMIAL ARE:

G0 : -0.10155089
G1 : 2.45448414
G2 : -7.79411937
G3 : 4.19004645
G4 : -2.45199927
G5 : -1.12298794
G6 : 0.27101442

VITA ²

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CYLINDRIC - CYLINDRIC CRANKS

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ical Engineering from Oklahoma State University in May, 1972. Re-
cieved the Masters of Science degree from Oklahoma State Univer-
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Professional Experience: Two months as engineering trainee at "El
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Non-Professional Organizations: Member of the Honorary Mechanical
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ing Society (ΣT), member of the Honorary Engineering Society
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Professional Organizations: Associate member of the American Society
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